

Introduction to WQO- and BQO-Theory

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Outline

1. Well-Quasi-Orders
2. Operations on WQO
3. Kruskal's Theorem

- For papers, slides, handout, and further references see <http://www.science.uva.nl/~ckissig/bqo/>

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4. Barriers and Blocks
5. Better-Quasi-Orders
6. Forerunning

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Definition (Initial (Final) Segments of Quasi-Orders) An *Initial (Final) Segment* of a quasi-ordered set Q is a downward (upward) closed subset $X \subseteq Q$, i.e. whenever $x \in X$, then $\forall y \leq_Q x. y \in X$ ($\forall x \leq_Q y. y \in X$).

Sequences

Definition (Sequences) A Q -Sequence is a function $f : \alpha \rightarrow Q$ for $\alpha \leq \omega$. A Q -sequence f is *infinite* if $\alpha = \omega$, and *finite* otherwise. Furthermore, a Q -sequence f is

- *good* if there are $i < j < \omega$ such that $f(i) \leq_Q f(j)$
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Well-Quasi-Orders

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Definition (Well-Quasi-Orders) \leq_Q is a *Well-Quasi-Order* on Q if there is no bad Q -sequence.

Equivalent Definitions of WQO

The following are equivalent

1. Q is Well-Quasi-Ordered
2. Q is Well-Founded and has no Infinite Antichains
3. Any Q -Sequence contains a Perfect Subsequence

↗ Higman, Ordering by divisibility in abstract algebras, 1952

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Equivalent Definitions of WQO

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5. Any subset $X \subseteq Q$ has a Finite Basis
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Equivalent Definitions of WQO

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5. Any subset $X \subseteq Q$ has a Finite Basis (Finite-Basis-Property)

Definition (Finite-Basis-Property) Given a set Q quasi-ordered by \leq_Q .

- For any subset $X \subseteq Q$, $cl(X) = \{q \in Q \mid \exists x \in X. x \leq_Q q\}$.
- A subset $X \subseteq Q$ is called *closed* iff $X = cl(X)$.
- Q has the *Finite-Basis-Property* iff any closed set in Q is the closure of a finite set.

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5. Any subset $X \subseteq Q$ has a Finite Basis
(Finite-Basis-Property)
6. There is no infinite strictly increasing Sequence of Final Segments of Q ordered by Inclusion
7. There is no infinite strictly descending Sequence of Initial Segments of Q ordered by Inclusion

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Minimal Bad Sequences

Definition (Minimal Bad Sequences) A bad sequence $f(0) \not\leq_Q f(1) \not\leq_Q \dots$ is called *Minimal Bad* iff whenever $g(0) \not\leq_Q g(1) \not\leq_Q \dots$ is a bad sequences, there is an index $i < \omega$ such that $f(j) = g(j)$ for $j < i$ and $f(i) \leq_Q g(i)$.

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Lemma (Hodkinson 1.6) If \leq_Q is well-founded but not wqo then there is a minimal bad sequence.

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Lemma (Hodkinson 1.6) If \leq_Q is well-founded but not wqo then there is a minimal bad sequence.

Lemma (Hodkinson 1.7) Let $f(0) \not\leq_Q f(1) \not\leq_Q \dots$ be minimal bad. Put $Y = \{q \in Q \mid q <_Q f(i) \text{ for some } i < \omega\}$. Then \leq_Q is a wqo on Y .

↗ Hodkinson, Kruskal's Theorem and Nash-Williams Theory, 2003

Operations

For Q quasi-ordered, the following are immediate

- If $Q' \subseteq Q$ and Q is WQO, then so is Q' .

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Theorem (Nash-William) If Q is WQO, then so is $Fin(Seq(Q))$, the class of Q -sequences of finite range.

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The following result is due to Rado

- If Q is WQO, then $P(Q)$ is not necessarily WQO.

↗ Milner, Basic WQO- and BQO-Theory, 1984

Kruskal's Theorem

Theorem (Kruskal) The collection of finite pointed Trees is a WQO.

↗ Hodkinson, Kruskal's theorem and Nash-Williams-theory, 2003

↗ Kruskal, Well-quasi-ordering, the tree theorem, and Vaszonyi's conjecture, 1960

Kruskal's Theorem

Theorem (Kruskal) The collection of finite pointed Trees is a WQO.

Proof following Hodges and Hodkinson

- Define the *Decent Embedding* f on pointed trees.
- Define \leq such that $T_1 \leq T_2$ iff there is a decent embedding $f : T_1 \rightarrow T_2$.
- Prove \leq a WQO.

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Motivation for BQOs

Operation	preserve WQO	preserve BQO
$<^\omega(-)$	yes (Higman)	
$Fin(Seq(-))$	yes (Nash-William)	
$Seq(-)$	<i>no</i>	
$[Q]^{<^\omega}$	yes	
$P(-)$	<i>no (Rado)</i>	
$T_{(-)}$	yes (Kruskal)	
$T_{(-)}^{\leq/\geq\omega}$	<i>no (Rado)</i>	

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Barriers

Definition (Blocks) Let $B \subseteq [Q]^{<\omega}$ be infinite, then B is a block on Q if for every infinite $X \subseteq Q$ there is an initial segment of X in B and for all $b_0, b_1 \in B$, $b_0 \not\prec b_1$.

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Example :

- $[\omega]^\alpha$ is a barrier for any $\alpha < \omega$

Results on Blocks and Barriers

Lemma () If B is a block and $B = B_0 \cup B_1$, then there is a block C such that $C \subseteq B_0$ and $C \subseteq B_1$.

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Lemma () If B is a block and $B = B_0 \cup B_1$, then there is a block C such that $C \subseteq B_0$ and $C \subseteq B_1$.

Lemma () Any block contains a barrier.

Patterns

Definition (Patterns) A Q -pattern is a function $f : B \rightarrow Q$ whose domain is a barrier. Such a Q -pattern f is called

- *good* if there are $b_1, b_2 \in B$ such that $b_1 \triangleleft b_2$ and $f(b_1) \leq_Q f(b_2)$
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Theorem () Any Q -pattern contains either a bad or a perfect sub-pattern.

Better-Quasi-Orders

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Definition (Better-Quasi-Orders) \leq_Q is a *Better-Quasi-Order* on Q if there is no bad Q -pattern.

Theorem () Q is wo $\rightarrow Q$ is wqo $\rightarrow Q$ is bqo.

Forerunning Technique

Definition (Forerunning (Barriers)) Given barriers B and C , we say B foreruns C , written $B \underline{\downarrow} C$, if

- $\bigcup C \subseteq \bigcup B$ and
- $\forall c \in C. \exists b \in B. b \preceq c$

B strictly foreruns C , written $B \downarrow C$, if additionally

- $\exists c \in C. \exists b \in B. b \prec c$, i.e. $C \not\subseteq B$

↗ Laver, Better-Quasi-Orderings and a Class of Trees, 1978

Forerunning Technique

Definition (Forerunning (Patterns)) Given a fixed rank function $\rho : Q \rightarrow On$ and Q -patterns $f : B \rightarrow Q$ and $g : C \rightarrow Q$, then we say f (strictly) foreruns g , written $(f \downarrow g)$ $f \underline{\downarrow} g$, if B (strictly) foreruns C and

- $g(b) = f(b)$ if $b \in B \cap C$
- $g(c) \leq_Q f(b)$ and $\rho(g(c)) < \rho(f(b))$ if $b \in B$ and $c \in C$ and $b \prec c$

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Definition (Minimal Bad Q -Pattern) A bad Q -pattern $f : B \rightarrow Q$ is *minimal bad* if for all bad Q -patterns $g : C \rightarrow Q$, $f \not\downarrow g$.

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Generalizing Higman's Theorem

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Theorem () If Q bqo then $Seq(Q)$ bqo.

Generalizing Higman's Theorem

Theorem (Laver) Let Q be a ranked bqo and suppose f is a bad Q -pattern, then there is a minimal bad Q -pattern g such that $f \underline{\downarrow} g$.

Theorem () If Q bqo then $Seq(Q)$ bqo.

Proof by contradiction

- Rank function ρ such that $\rho(s) = \alpha$ if $s \in {}^\alpha Q$
- Suppose minimal bad $Seq(Q)$ -pattern $f : B \rightarrow Seq(Q)$
- Pick sub-barrier $C \subseteq B$ and distinguish for all $c \in C$
 - a) $\rho(f(c)) = 1$
 - b) $\rho(f(c)) = \beta + 1$
 - c) $\rho(f(c)) = \lambda$ is a limit ordinal(a) is impossible, for (b) and (c) contradict minimality of f

End of our talks

Thanks for *your attention...*