Introduction to WQO and BQO Theory

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1 Notation

- Let $\leq_Q$ is a quasi-order, then
  - $x =_Q y$ iff $x \leq_Q y$ and $y \leq_Q x$
  - $x <_Q y$ iff $y \leq_Q y$ and $x \neq_Q y$
  - $x||_Q y$ iff $x \not\leq_Q y$ and $y \not\leq_Q x$
- ${}^{\alpha}Q$, the family of all Q-sequences $f : \alpha \to Q$ of length α
- ${}^{<\alpha}Q = \bigcup^{\beta}_{\beta<\alpha} Q$
- $Seq(Q) = \bigcup^{\beta}_{\alpha \in On} Q$
- $Fin(S)$, the set of finite sequences from S
- $[Q]^{<\omega}$, the set of finite subsets of Q.
- T_Q, the family of Q-labelled trees
- For a finite Q-sequence s, ${}_*s$ (s_*) is s with the first (last) term deleted
- For Q-sequences s and t, $s \preceq t$ iff s is an initial segment of t ($s = t \restriction \alpha$)
- $s \prec t$ iff $s \preceq t$ and $s \neq t$
- $s \triangleleft t$ iff ${}_*s \prec t$
- For B, a barrier, $B(2) = \{b_1 \cup b_2 \mid b_1, b_2 \in B \wedge b_1 \triangleleft b_2\}$
- For barriers B and C, ($B \downarrow C$) $B \underline{\downarrow} C$ if B (strictly) foreruns C
- For Q-patterns f and g, ($B \downarrow C$) $f \underline{\downarrow} g$ if f (strictly) foreruns g

2 Definitions

Definition 1 (Quasi-Order) $(X, \leq)$ *is a* Quasi-Ordering *if* $\leq$ *is reflexive on* X *and transitive.*

Definition 2 (Well-founded Quasi-Orders) *A Quasi-Order* $\leq_Q$ *is* well-founded *iff there is no infinitely descending chain* $q_0 >_Q q_1 >_Q \ldots$.

Definition 3 (Initial (Final) Segments of Quasi-Orders) *An* Initial (Final) Segment *of a quasi-ordered set Q is a downward (upward) closed subset $X \subseteq Q$, i.e. whenever $x \in X$, then $\forall y \leq_Q x.y \in X$ ($\forall x \leq_Q y.y \in X$).*

Definition 4 (Sequences) *A Q-*Sequence *is a function $f : \alpha \to Q$ for $\alpha \leq \omega$. A Q-sequence f is* infinite *if $\alpha = \omega$, and* finite *otherwise. Furthermore, a Q-sequence f is*

- good *if there are $i < j < \omega$ such that $f(i) \leq_Q f(j)$*
- bad *if it is not good*
- perfect *if $i \leq j$ implies $f(i) \leq_Q f(j)$*

Definition 5 (Minimal Bad Sequences) *A bad sequence $f(0) \not\leq_Q f(1) \not\leq_Q \ldots$ is called* Minimal Bad *iff whenever $g(0) \not\leq_Q g(1) \not\leq_Q \ldots$ is a bad sequences, there is an index $i < \omega$ such that $f(j) = g(j)$ for $j < i$ and $f(i) \leq_Q g(i)$.*

Definition 6 (Well-Quasi-Orders) *$\leq_Q$ is a* Well-Quasi-Order *on Q if there is no bad Q-sequence.*

Definition 7 (Finite-Basis-Property) *Given a set Q quasi-ordered by $\leq_Q$.*

- *For any subset $X \subseteq Q$, $cl(X) = \{q \in Q \mid \exists x \in X.x \leq_Q q\}$.*
- *A subset $X \subseteq Q$ is called* closed *iff $X = cl(X)$.*
- *Q has the* Finite-Basis-Property *iff any closed set in Q is the closure of a finite set.*

Definition 8 (Blocks) *Let $B \subseteq [Q]^{<\omega}$ be infinite, then B is a block on Q if for every infinite $X \subseteq Q$ there is an initial segment of X in B and for all $b_0, b_1 \in B$, $b_0 \not\prec b_1$.*

Definition 9 (Barriers) *A block B on a set Q is a barrier iff B is an antichain w.r.t. $\subseteq$.*

Definition 10 (Better-Quasi-Orders) *$\leq_Q$ is a* Better-Quasi-Order *on Q if there is no bad Q-pattern.*

Definition 11 (Forerunning) *Given barriers B and C, we says B* foreruns *C, written $B \underline{\downarrow} C$, if*

- *$\bigcup C \subseteq \bigcup B$ and*
- *$\forall c \in C.\exists b \in B.b \preceq c$*

B strictly foreruns *C ($B \downarrow C$) if additionally*

- *$\exists c \in C.\exists b \in B.b \prec c$, i.e. $C \not\subseteq B$*

Given a fixed rank function $\rho : Q \to On$ and Q-patterns $f : B \to Q$ and $g : C \to Q$, then we say f (strictly) foreruns *g, written ($f \downarrow g$) $f \underline{\downarrow} g$, if ($B \downarrow C$) $B \underline{\downarrow} C$ and*

- *$g(b) = f(b)$ if $b \in B \cap C$*
- *$g(c) \leq_Q f(b)$ and $\rho(g(c)) < \rho(f(b)$ if $b \in B$ and $c \in C$ and $b \prec c$*

3 Operations, and the order they preserve

Operation	preserve WQO	preserve BQO
subsets	yes (Milner 1.3)	
finite union	yes (Milner 1.4)	yes (Milner 2.13)
finite product	yes (Milner 1.5)	yes (Milner 2.15)
${}^{<\omega}(-)$	yes (Higman's Theorem)	
$Fin(Seq(-))$	yes (Nash-William)	
$Seq(-)$	no	yes (Milner 2.22)
$[Q]^{<\omega}$	yes (Milner 1.7)	
$P(-)$	no (Rado)	yes (Milner 2.1)
$T_{(-)}$	yes (Kruskal's Theorem)	yes (Nash-William's Theorem)
$T_{(-)}^{\leq/\geq\omega}$	no (Rado)	yes (Nash-William's Theorem)