TQFTs and Invariants of 3-Manifolds

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Contents

Chapter 1

Introduction

1.1 Summary

In this tutorial we discuss invariants of 3-manifolds (with banded links embedded), and how to compute them by means of 3-dimensional topological quantum field theories (3D-TQFTs). The 3D-TQFTs, we consider, shall be defined in the Reshetikhin-Turaev construction[6] from a modular category $\mathcal{V}$ over a ground ring K as modular functors from (subcategories of) the category $3Cob$ of 3-cobordisms into categories of projective modules over K.

The 3-cobordisms have embedded ribbon links. Modularity will use that these ribbon links Ω meet the boundary ∂M of the cobordisms in some component Σ, such that the bands are incident with arcs decorating surfaces. Such a cobordism can be closed by gluing handlebodies with embedded bands. Modularity, decoration with arcs, and gluing makes it necessary that we consider ribbon graphs instead. These consist of annuli (non-M\"obius ribbon knots), bands, and coupons which are a ribbon-equivalent of nodes in graphs.

The ribbon graphs will be coloured in a modular category $\mathcal{V}$. The colouring carries over to the decoration of surfaces and to a form of gluing compatible with decoration and colouring. It turns out that a cobordism in such a cobordism theory with decorations is completely described by the topological properties of the underlying manifold and the decorations. In order to define modular functors from cobordism theories with decorations, it suffices to look at the decorations in the surfaces.

The constructed 3D-TQFT is non-degenerate, but has anomalies. In fact the axiomatic characterisation of 3D-TQFT from modular functors suggest that anomalies are introduced by gluing. An additional parameterisation in weights in an abelian group G (basically, of the invertible elements of the ground ring) yields a standard renormalisation, which moves anomalies into gluing and produces a cobordism theory with decorations and weights. The anomalies can often be computed by 2-cocycles valued in G. In particular our construction yields anomaly 2-cocycles, that we can compute by Maslov-indices of Lagrangian subspaces of the homologies on the surfaces.

1.2 Historical Background

Modular functors arose from 2-dimensional rational conformal field theory. Segal[7], Moore and Seiberg[5] first gave an axiomatic definition. Topological quantum field theories were inspired by the work of Witten[10, 10]. Unitary TQFTs were first axiomatised by Atiyah[1] in an extension of Segal's modular functors. Quinn has first studied the axiomatic foundations of TQFTs in an abstract categorical setting. Domain categories took the place of cobordism theories.

1.3 Organisation of These Notes

1.3.1 Contents and Dependencies

In Section 2 we introduce the foundational concepts in monoidal categories, modular categories and functors, links and graphs. These foundations allow us to define 3-dimensional topological quantum field theories in Chapter 3 axiomatically. Chapter 4 will be concerned with the concrete construction of a 3-dimensional topological quantum field theory. The constructed TQFT is non-degenerate but has anomalies. We look into the anomalies in Chapter 5, and propose methods to renormalise TQFTs. The so constructed 3-dimensional TQFTs will be in bijection with the quantum invariants of 3-manifolds with embedded banded links.

Dependencies between the chapters are linear. The knowledgeable reader may skip any of the preliminary sections and return for reference later without harm.

1.3.2 Background

These notes are largely based on Chapters I-V of [8], and on the lecture series on "Integral TQFT" by Masbaum held at Copenhagen in 2007. These notes are prepared for an invited seminar at the University of Warsaw on 27 February 2010. The material on homologies on surfaces stems mostly from [2].

1.4 Apologies and Request for Comments

I understand that the lack of images renders the presentation almost illegible. I plan to add figures in later versions of these notes. My apologies for this deficite. Nevertheless, I welcome comments and suggestions at `tqft@christiankissig.de`. Additional material, references, and updated notes for this seminar can be found at http://www.christiankissig.de/3dtqft.

Chapter 2

Preliminaries

2.1 Monoidal Categories

Monoidal categories are regular categories endowed with a monoidal structure.

Definition 1. A *monoidal category* $\mathcal{V} := \langle V, \otimes, \mathbb{I} \rangle$ consist of a regular category V with a 2-multi-endofunctor with identity object so that there are natural isomorphisms $l_X : \mathbb{I} \otimes X \to X$ and $r_X : X \otimes \mathbb{I} \to X$, subject to identity and associativity laws

A monoidal category may be *symmetric*, if it comes with a natural isomorphism $s : (-) \otimes (+) \cong (+) \otimes (-)$, and *closed* if $\mathcal{V}(X, -)$ is right adjoint to $(-) \otimes X$ for all objects X of $\mathcal{V}$.

Furthermore, monoidal categories may be braided.

Definition 2. A *braiding* in a monoidal category $\mathcal{V}$ is a natural isomorphism $c : (-) \otimes (+) \Rightarrow (+) \otimes (-)$ such that

$$c_{X,Y \otimes Z} = (id_X \otimes c_{X,Z})(c_{X,Y} \otimes id_Z) \tag{2.1}$$

$$c_{X \otimes Y,Z} = (c_{X,Z} \otimes id_V)(id_X \otimes c_{V,W}) \tag{2.2}$$

A twist in a braided monoidal category $\mathcal{V}$ is a natural isomorphism $t : (-) \Rightarrow (-)$ such that

$$t_{X \otimes Y} = c_{Y,X} c_{X,Y}(t_X \otimes t_Y) \tag{2.3}$$

Example 3. Categories of modules are symmetric monoidal closed under $\otimes$ being the coproduct $\coprod$ (disjoint union).

Example 4. Categories of 3-cobordisms are monoidal under disjoint union $\coprod$ and the empty cobordism $\emptyset$.

Dualities generalise non-degenerate bilinear forms to monoidal categories.

Definition 5. A duality on a monoidal category $\mathcal{V}$ consists of an endomorphism $(-)^*$on the objects of $\mathcal{V}$ and natural transformations $b : (-) \Rightarrow (-) \otimes (-)^*$ and $d : (-)^* \otimes (-) \Rightarrow (-)$ such that

$$(id_X \otimes d_X)(b_X \otimes id_X) = id_X \tag{2.4}$$

$$(d_X \otimes id_{X^*})(id_{X^*} \otimes b_X) = id_{X^*} \tag{2.5}$$

Naturality of b and d accounts for bilinearity, the rest of the analogy we leave to the reader.

Remark 6. There are elegant graphical calculi for reasoning in monoidal categories. See for instance Selinger "A survey of graphical languages for monoidal categories" for various kinds of monoidal categories or Turaev "Quantum Invariants of Knots and 3-Manifolds" Section I.1.6 for ribbon categories in particular.

2.1.1 Ribbon Categories

Definition 7. A ribbon category is a monoidal category $\mathcal{V}$ with braiding c, twist t, and compatible duality $((-)^*, b, d)$. A ribbon category is strict if its underlying monoidal category is strict.

Example 8. The category $Proj(K)$ of projective modules for a ring K is a ribbon category.

Example 9. Given a multiplicative abelian group G , a commutative ring K with unit, a bilinear pairing $c : G \times G \to K^*$where $K^*$is the multiplicative group of invertible elements of K. Bilinearity means that $c(gg', h) = c(g, h)c(g', h)$ and $c(g, hh') = c(g, h)c(g, h')$. We construct a ribbon category $\mathcal{V}$ with objects being the elements of G, endomorphisms $g \to g$ being elements of K, and homomorphisms $g \to h$ being zero. Composition is given by the multiplication in K, the tensor by multiplication in G.

Exercise 10. Define braiding and twist for $\mathcal{V}$ in Example 9.

Exercise 11. Verify the axioms of ribbon categories on the category $\mathcal{V}$ in Example 9.

Example 12. Finite-dimensional representations of quantum groups form a ribbon category. For a detailed discussion we refer the reader to [3].

Remark 13. MacLane's coherence theorem specialises to ribbon categories.

2.1.2 *Ab*-Categories

Definition 14. A category $\mathcal{V}$ is an *Ab*-category if each homset $Hom(X, Y)$ comes with the structure of an additive abelian group.

Exercise 15. Verify that homsets $End(X) = Hom(X, X)$ of endomorphisms in *Ab*-categories are rings.

Exercise 16. Verify that homsets $Hom(X, Y)$ in monoidal *Ab*-categories have the structure of left K-modules where $K = End(\mathbb{I})$ under $k \cdot f = k \otimes f$.

Proposition 17. *In ribbon Ab-categories multiplication with K is bilinear, in particular $k \otimes id_V = id_V \otimes k$ for all objects V of $\mathcal{V}$.*

Example 18. The category $Proj(K)$ of projective modules for a ring K is a ribbon *Ab*-category.

Example 19. The category $\mathcal{V}$ of Example 9 is an *Ab*-category.

Definition 20. An object V of $\mathcal{V}$ is simple if $k \mapsto k \otimes id_V$ defines a bijection $K \cong End(V)$.

Example 21. $\mathbb{I}$ is simple.

Definition 22. The trace $tr(f)$ of an endomorphism $f : V \to V$ on an object V of $\mathcal{V}$ is defined as

$$tr(f) = d_V c_{V,V^*}((t_V f) \otimes id_{V^*})b_V : \mathbb{I} \to \mathbb{I} \tag{2.6}$$

Definition 23. The dimension $dim(V)$ of an object V is defined as the trace of the identity morphism id_V, viz

$$dim(V) = tr(id_V) = d_V c_{V,V^*}(t \otimes id_{V^*})b_V : \mathbb{I} \to \mathbb{I} \tag{2.7}$$

Remark 24. As $\mathbb{I}$ is simple, dimension and trace are elements of the ground ring K.

2.1.3 Modular Categories

A decomposition in direct sums of indecomposables is known to exist for artinian, noetherian or semi-simple modules. General ribbon categories do not admit direct sums of objects, but we obtain decomposition of their identity morphisms via the notion of domination:

Definition 25. An object V of $\mathcal{V}$ is said to be dominated by a family $\{V_i\}_{i\in I}$ of objects of $\mathcal{V}$ if id_V decomposes into the sum $\sum_{j\in J} f_{w(j)}g_{w(j)}$ of a finite zig-zag of morphisms $f_{w(j)} : V_{w(j)} \to V$ and $g_{w(j)} : V \to V_{w(j)}$ indexed by a finite word$w \in I^*$.

Exercise 26. Show that if $\mathcal{V}$ admits direct sums, then an object V of $\mathcal{V}$ is dominated by a family of objects of $\mathcal{V}$ iff there is an object W of $\mathcal{V}$ such that $V \oplus W$ decomposes into a finite direct sum of objects of this category.

Definition 27. A modular category is a ribbon *Ab*-category $\mathcal{V}$ with a finite family $\mathbb{V}$ =$\{V_i\}_{i\in I}$ of objects of $\mathcal{V}$ satisfying

1. (normalisation) $\mathbb{I} \in \mathbb{V}$
2. (duality) $\mathbb{V}$ is closed under $(-)^*$.
3. (domination) Each object of $\mathcal{V}$ is dominated by $\mathbb{V}$.
4. (non-degeneracy) The square matrix $[S_{ij}]_{i,j\in I}$ is invertible over the ground ring K of $\mathcal{V}$.

$$S_{ij} = tr(c_{V_j,V_i} \circ c_{V_i,V_j}) \tag{2.8}$$

Exercise 28. Using the (non-degeneracy) axiom show that $\mathbb{I}$ appears precisely once in $\mathbb{V}$.

Remark 29. Semi-simple ribbon categories generalise modular categories in that they allow arbitrary families $\mathbb{V} = \{V_i\}_{i\in I}$. Schur's axiom asserts$\mathbb{V}$ to be discrete in the sense that $Hom(V_i, V_j) = 0$ for all $i, j \in I$ with $i \neq j$. Semi-simple ribbon categories give rise to topological invariants of links and ribbon graphs in $\mathbb{R}^3$, but not 3-manifolds.

2.1.4 The Ground Ring of a Modular Category

Let $\mathcal{V}$ be a modular category.

Definition 30. The rank of $\mathcal{V}$ is an element $rank(\mathcal{V}) \in K$ of the ground ring $K = End(\mathbb{I})$ of $\mathcal{V}$, such that

$$rank(\mathcal{V}) = \sum_{i \in I} (dim(i))^2 \tag{2.9}$$

The rank is not unique, in particular $-rank(\mathcal{V})$ is a rank of $\mathcal{V}$.

Definition 31. $\Delta \in K$ is uniquely determined by

$$\Delta = \sum_{i \in I} v_i^{-1} (dim(i))^2 \tag{2.10}$$

2.2 Ribbon Graphs

2.2.1 Ribbon Graphs, Uncoloured

Definition 32. A ribbon graph Ω embedded in a manifold M is a compact oriented surface composite of a finite set of

1. bands, that are homeomorphic images of $[0,1] \times [0,1]$ where we call $[0,1] \times 0$ and $[0,1] \times 1$ the bases of the band
2. coupons, that are bands with one base distinguished as their top base, and one as their bottom base
3. and annuli, that are homeomorphic images of cylinders $S^1 \times [0,1]$ (Informally, these are knots that are not Moebius strips.)

Remark 33. Choosing an orientation of a ribbon graph is equivalent to choosing a preferred side.

2.2.2 Ribbon Graphs, Coloured

Definition 34. In this text we consider ribbon graphs that are coloured in a modular category $\mathcal{V}$ with duality. Bands and annuli are coloured with objects of $\mathcal{V}$, and coupons are coloured with morphisms

$$f : V_1^{d_0} \otimes ... \otimes V_m^{d_m} \to W_1^{e_1} \otimes ... \otimes W_n^{e_n} \tag{2.11}$$

where $V_1, ..., V_m$ are the colours and $d_1, ..., d_m$ are the directions (in- or outgoing) of the bands incident to the top edge, $W_1, ..., W_n$ are the colours and $e_1, ..., e_n$are the directions of the bands incident to the bottom edge. Where $V^{+1} = V$ and $V^{-1} = V^*$ and similar for W.

Example 35. The category $Rib_{\mathcal{V}}$ of $\mathcal{V}$-coloured ribbon graphs has finite squences $\langle\langle V_1, v_1\rangle, ..., \langle V_m, v_m\rangle\rangle$ of pairs of coloursV_i and signs $v_i \in -1, +1$ as objects, and isotopy types of $\mathcal{V}$-coloured ribbon graphs as morphisms with colours and directions at top and bottom edge incident with the respective object.

The following is a fundamental observation for the definition of isotopy invariants of coloured ribbon graphs, topological invariants of 3-manifolds with embedded ribbon graphs, and later on for the definition of quantum invariants of 3-cobordisms. F from the its action on the basic directed components of ribbon graphs, overcrossing X, loop ϕ, cup $\bigcup$, and cap $\bigcap$.

Definition 36. There is a unique functor $F : Rib_{\mathcal{V}} \to \mathcal{V}$ taking

- objects $\langle V, +1\rangle$ to V, and $\langle V, -1\rangle$to V^*
- for colours V and W, $F(X^+_{V,W}) = c_{V,W}$, $F(\phi_V) = \theta_V$, $F(\bigcup_V) = b_V$, and $F(\bigcap_V) = d_V$ where $X^+$is a positive overcrossing, $\phi$is a positive loop, $\bigcap$ is a cap directed left-to-right, $\bigcup$ is a cup directed left-to-right
- a $\mathcal{V}$-coloured graph with one coupon to the colour of the coupon

2.3 Manifolds

By manifolds we mean compact oriented manifold M with boundary ∂M. A manifolds with $\partial M = \emptyset$ is called closed.

2.3.1 Collars

A manifold M is said to have collars if it is homeomorphic to a copy of itself with a cylinder $B^1 \times [0, 1]$ glued to each component Σ of the boundary ∂M.

2.3.2 Surgery on Links in S^3

Let L be a framed link in S^3 with m components, $L_1, ..., L_m$. A closed regular neighbourhood U of L consists of m disjoint solid tori, $U_1,..., U_m$, whose cores are the components of L. Removing the solid tori from S^3 yields a remainder R, one can define a bijection f between ∂R and $S^3$and glues R and S^3back together along f.

Remark 37. Note that in the context of differentiable manifolds surgery along embedded links in S^3 creates Wilson loops.

Theorem 38. *Every closed connected oriented 3-manifold M is obtained by surgery on S^3along a framed link L.*

This theorem is due to Lickorish[4] and Wallace[9]. The proof uses the theorem of Rokhlin that every closed oriented 3-manifold bounds a compact oriented piecewise-linear 4-manifold.

2.3.3 Topological Invariants of 3-Manifolds

Topological invariants of 3-manifolds are topological properties invariant under homeomorphisms. Here we consider a construction such invariants for 3-manifolds with embedded ribbon graphs coloured in modular category $\mathcal{V}$, such that braiding, twist, and duality reflect the basic components of ribbon graphs.

Roughly we obtain topological invarians of 3-manifolds as follows. Let M be a closed connected oriented 3-manifold. By Theorem 38 M is obtained by surgery on S^3along a framed link L. Let L have components $L_1, ..., L_m$. Fix an orientation of L, and let $col(L)$ denote the finite set of all possible colourings of the components of L in a set I. Fixing a colouring λ, we obtain a ribbon graph $\Gamma(L, \lambda)$ without loose ends. Define

$$\{L\}_\lambda = dim(\lambda) F(\Gamma(L, \lambda)) \text{ where } dim(\lambda) = \prod_{i=1}^{m} dim(\lambda(L_i)) \tag{2.12}$$

$$\text{and } \{L\} = \sum_{\lambda \in col(L)} \{L\}_\lambda \tag{2.13}$$

Remark 39. $\{L\}$ does not depend on the choice of orientation of L.

Then define

$$\tau(M) = \Delta^{\sigma(L)} (rank\{L\})^{-\sigma(L)-m-1} \tag{2.14}$$

Theorem 40. *Let M be a 3-manifold, then $\tau(M)$ is a topological invariant of M.*

2.3.4 Examples of Manifolds

1. S1 is the unit circle. S1 bounds the unit disc B2, $S^1 = \partial B^2$
2. The 3-sphere S3 is 4-dimensional hypersphere. S3 bounds the closed 4-ball B4, $S^3 = \partial B^4$.

Chapter 3

Elements of 3D-Topological Quantum Field Theories

3.1 Cobordisms, Gluing Patterns, Cobordism Theories

Cobordisms are 3-manifolds in which we distinguish in- and out-boundary.

Definition 41. A *3-cobordism* is a 3-manifold M whose boundary ∂M is partitioned into $\partial_- M$ (in-boundary) and $\partial_+ M$ (out-boundary).

Example 42. Let X be a closed oriented surface, the cylinder $X \times [0,1]$ is a 3-cobordism with boundaries $\partial_-(X \times [0,1]) = -X$ and $\partial_+(X \times [0,1]) = X$. $-X$ is X with reversed orientation.

Cobordisms may be glued along a parameterising homeomorphism. Points from M and N coincide in the resulting cobordism iff they are mapped onto each other by the parameterising morphism. We refer to the standard literature to a more detailed account of gluing.

Definition 43. Let $\langle M, \partial_- M, \partial_+ M\rangle$and $\langle N, \partial_- N, \partial_+ N\rangle$ be 3-cobordisms, and let $f : \partial_+ M \to \partial_- N$ be a homeomorphism. The result of gluing M and N along f is a 3-cobordism $\langle M \cup_f N, \partial_- M, \partial_+ N\rangle$ in which points $m \in M$ and $n \in N$ coincide iff $n = f(m)$. The triple $\langle M, N, f\rangle$is called a *gluing pattern.* A homeomorphism of gluing patterns $\langle M, N, f\rangle \to \langle M', N', f'\rangle$consists of homeomorphisms $h : M \to M'$and $g : N \to N'$ preserving bases such that $f'g \mid_{\partial_+ M} = hf : \partial_+ M \to \partial_- N'$.

Definition 44. 3-cobordisms and their surfaces form a (monoidal) category, $3Cob$, such that gluing as composition and cylinders as identity morphisms. Disjoint union, $\coprod$, and empty cobordism and surface, $\emptyset$, provide the monoidal structure. By surface we mean surfaces of 3-cobordisms.

Definition 45. The cobordism theories $\mathbb{C}$, we regard in these notes, are subcategories of $3Cob$ closed under cylinders, gluing, disjoint union, and empty cobordisms and surfaces.

Example 46. In particular, $3Cob$ is a cobordism theory.

3.2 2-Dimensional Modular Functors

Definition 47. A *2-dimensional modular functor* for a ground ring K is a monoidal functor from the category of 2-dimensional surfaces with homeomorphisms to the category of projective K-modules and K-modules isomorphisms.

3.3 3-Dimensional Topological Quantum Field Theories

TQFTs extend 2-dimensional modular functors by maps assigning to 3-cobordisms operator invariants.

3.3.1 Axiomatic Definition of 3D-TQFTs

We introduce 3D-TQFTs axiomatically.

Definition 48. A *3-dimensional topological quantum field theory* (3D-TQFT) on a 3-cobordism theory $\mathbb{C}$ consists of a modular functor $\mathcal{T}$ with ground ring K and a map τ assigning to every cobordism M a K-module homomorphism $\tau(M) : \mathcal{T}\partial_- M \to \mathcal{T}\partial_+ M$, the operator invariant of M.

1. (naturality) Let $M_1$and M_2 be cobordisms of $\mathbb{C}$ and $f : M_1 \to M_2$be a homeomorphism preserving their bases, then $\mathcal{T}(f \mid_{\partial_+(M_1)}) \circ \tau(M_1) = \tau(M_2) \circ \mathcal{T}(f \mid_{\partial_-(M_1)})$ commutes.

2. (multiplicativity) Let M be the disjoint union of cobordisms M_1 and M_2 of $\mathbb{C}$ then $\tau(M) = \tau(M_1) \otimes \tau(M_2)$.

3. (functoriality) Let M be obtained by gluing cobordisms M_1 and M_2 of $\mathbb{C}$ along a homeomorphism $f : \partial_+(M_1) \to \partial_-(M_2)$ then there is an invertible $k \in K$, such that

$$\tau(M) = k\tau(M_2) \circ (\mathcal{T} f) \circ \tau(M_1) \tag{3.1}$$

4. (normalisation) For any closed oriented surface X,

$$\tau(X \times [0, 1], X \times 0, X \times 1) = id_{\mathcal{T}(X)} \tag{3.2}$$

Remark 49. The weight-parameter $k \in K$ spoils τ being a monoidal functor. We may recover a functorial behaviour by renormalising the cobordism theory in Chapter 5. This situation indicates anomalies.

Example 50. Let $\mathcal{T}$ be the modular functor restricted to finite cell spaces, and fix an invertible element $k \in K$. For any finite cell triple $\langle M, X, Y \rangle$, define the operator invariant $\tau\langle M, X, Y \rangle := k^{\chi(M,X)}$ where χ computes the Euler characteristic. Then $\langle \mathcal{T}, \tau \rangle$ is an anomaly-free TQFT.

Example 51. Fix an integer $i \geq 0$ and a finite abelian group G whose order is invertible in K. For any finite cell-space X, set $\mathcal{T}(X) = K[H_i(X; G)]$. Thus $\mathcal{T}(X)$ is a module of formal linear combinations of elements of the homology $H_i(X; G)$ with coefficients in K. Note that $\mathcal{T}(\emptyset) = K[H_i(X; G)] = K[0] =$

K. The action of cell homeomorphisms is induced by their action in H_i. The operator invariant $\tau\langle M, X, Y\rangle$of a finite cell-triple carries any $g \in H_i(X; G)$ into the formal sum of all those $h \in H_i(X; G)$ homological to g in M. This yields a TQFT with non-trivial anomalies, that we inspect in Chapter 5.

Exercise 52. Verify the axioms of TQFTs in Examples 50 and 51.

Definition 53. A cobordism theory $\mathbb{C}$ is said to be involutive if we have an involution on the cobordisms M in $\mathbb{C}$, and the boundaries of cobordisms commute with negation, $-\partial M = \partial(-M)$. An involutive cobordism theory gives rise to a dual TQFT such that $\mathcal{T}^*(M) = (\mathcal{T}(M))^*$and $\tau^*(M) = (\tau(M))^*$.

Exercise 54. Show that the dual of a TQFT is a TQFT, in particular that $\tau^*(M) = \tau(-M)$ where $(\mathcal{T}(\partial M))^*$ and $T(-\partial M)$ are identified via $d^{\mathcal{T}}_{\partial M}$.

3.3.2 Homomorphisms and Isomorphisms of 3D-TQFTs, Non-Degenerate TQFTs

Definition 55. A homomorphism$\langle \mathcal{T}_1, \tau_1\rangle \Rightarrow \langle \mathcal{T}_2, \tau_2\rangle$ between TQFTs on the same 3-cobordism theory $\mathbb{C}$ and over the same ground ring K is a natural transformation between the underlying modular functors. Isomorphisms $\langle \mathcal{T}_1, \tau_1\rangle \cong \langle \mathcal{T}_2, \tau_2\rangle$are respectively natural isomorphisms $\mathcal{T}_1 \cong \mathcal{T}_2$.

Example 56. The identity natural transformation $\mathcal{T} \Rightarrow \mathcal{T}$ extends to an isomorphism $\langle \mathcal{T}, \tau\rangle \cong \langle \mathcal{T}, \tau\rangle$.

The following theorem implies that operator invariants of isomorphic 3D-TQFTs coincide.

Lemma 57. *Let $g : \langle \mathcal{T}_1, \tau_1\rangle \to \langle \mathcal{T}_2, \tau_2\rangle$be an isomorphism of 3D-TQFTs , then $g(\emptyset) = id_K$ and for any 3-cobordism M we have $\tau_2(M) = g(\partial M)(\tau_1(M)) \in \mathcal{T}_2(\partial M)$.*

Definition 58. A TQFT $\langle \mathcal{T}, \tau\rangle$ is called *non-degenerate* if for any closed oriented surface X, $\{\tau\langle M, f\rangle \in \mathcal{T}(X)\}_{\langle M,f\rangle}$ generate the module $\mathcal{T}(X)$ over the ground ring K, where $\langle M, f\rangle$runs over all 3-cobordisms M with X-parameterisation f.

Corollary 59. *It follows from Lemma 57 that non-degeneracy is invariant under isomorphisms.*

The following theorem reduces the isomorphism problem of TQFTs to a simpler problem of K-valued functions - of the operator invariants.

Theorem 60. *Anomaly-free non-degenerate TQFTs $\langle \mathcal{T}_1, \tau_1\rangle$ and $\langle \mathcal{T}_2, \tau_2\rangle$based on the same ground ring K and the same cobordism theory $\mathbb{C}$ are isomorphic if $\tau_1(M) = \tau_2(M)$ for all cobordisms M.*

The proof of this theorem uses two constructions of modules for surfaces X, $\alpha_\tau(X)$ and $\beta_\tau(X)$. $\alpha_\tau(X)$ is the submodule of $\mathcal{T}(X)$ generated by the set $\{\tau(M, f)\}_{(M,f)}$ over all cobordisms M with X-parameterised (f) boundary. Then β_τ yields the quotient module

$$\beta_\tau(X) = \alpha_\tau(X)/(\alpha_\tau(X) \cap Ann(\alpha_\tau(-X))) \tag{3.3}$$

where $Ann(-)$ computes the set of annihilators with respect to the pairing $d^{\tau}_X : \mathcal{T}(X) \otimes_K \mathcal{T}(-X) \to K$.

If K has zero-characteristic we refine the above theorem, such that only one of the TQFTs must be non-degenerate.

Theorem 61. *Anomaly-free 3D-TQFTs $\langle \mathcal{T}_1, \tau_1 \rangle$ and $\langle \mathcal{T}_2, \tau_2 \rangle$ (one of them, non-degenerate) with ground ring K of zero characteristic (non-zero in the additive part) are isomorphic if $\tau_1(M) = \tau_2(M)$ for all 3-cobordisms M.*

Proof. □

Exercise 62. Verify that the dual of a non-degenerate TQFT for an involutive cobordism theory $\mathbb{C}$ is non-degenerate.

3.4 Quantum Invariants

We can glue 3-cobordisms M and N along a surface X, if M and N come with parametrising maps, that are homeomorphisms $f : X \to \partial M$ and $g : X \to \partial N$. We call $\langle M, f \rangle$ and $\langle N, g \rangle$ cobordisms with X-parametrized boundary. Then set

$$\tau^X \langle M, f \rangle := (\mathcal{T} f)^{-1} \circ (\tau(M)) \in \mathcal{T}(X) \tag{3.4}$$

Splitting systems allow us to look at 3-cobordisms as composites of finite families of 3-cobordisms. A quantum invariant τ_0 is said to have a splitting system, if it commutes suitably with the composite structure.

Definition 63. A splitting system for a 2-manifold X is a finite family of triples $\{\langle k_i, M_i, N_i \rangle \mid i \in I\}$ consisting of

- an element$k_i \in K$ of the ground ring,
- a 3-cobordism M_i with X-parameterised boundary and collar, and
- a 3-cobordism N_i with $(-X)$-parameterised boundary and collar,

such that for all cobordisms M and N of respectively X- and $(-X)$-parameterised boundary,

$$\tau_0(M \cup_X N) = \sum_{i \in I} k_i \tau_0(M \cup_X N_i) \tau_0(M_i \cup_X N) \tag{3.5}$$

A quantum invariant τ_0 is said to preserve splitting systems, if the splitting structure is compatible with the monoidal composition in $3Cob$. This is part of the definition of quantum invariants:

Definition 64. A quantum invariant is a K-valued homeomorphism invariant $\tau_0$of closed 3-manifolds, satisfying

1. (normalisation) $\tau_0(\emptyset) = \mathbb{I}$
2. (multiplicativity) $\tau_0(M \coprod N) = \tau_0(M)\tau_0(N)$
3. (splitting) $\tau_0$preserves splitting systems

The following is immediate from the definition of TQFT isomorphisms above.

Proposition 65. *The quantum invariants of* $3Cob$ *classify TQFTs up to isomorphism.*

Lemma 66. *Let* $\langle \mathcal{T}, \tau \rangle$ *be a TQFT based on* $3Cob$*, the function* τ *on the class of closed 3-cobordisms is a quantum invariant.*

Proof. Axioms (normalisation) and (multiplicativity) of Definition 64 are immediate from the definition of TQFTs. It remains to verify the (splitting) axiom. □

Theorem 67. *Every quantum invariant* τ_0 *of closed 3-cobordisms extends to a non-degenerate anomaly-free TQFT based on* $3Cob$*.*

Proof. Let $\mathcal{T}$ assign to a closed oriented surface X the free K-module generated by 3-cobordisms with X-parameterised boundary quotiented by Ann_X, the left annihilators of the bilinear pairing $w(M, N) = \tau_0(M \cup_X N)$. τ extends $\tau_0$such that for any 3-cobordism M with X-parameterised boundary, $\tau(M)$ takes a generator N of $\mathcal{T}(X)$ to result of gluing $N \cup_X M$. □

Theorem 68. *There is a bijective correspondence between isomorphism classes of non-degenerate anomaly-free TQFTs (based on* $3Cob$*) and quantum invariants of closed 3-cobordisms.*

Chapter 4

A Construction of a 3-Dimensional Topological Quantum Field Theory

In this section we construct a preliminary 3D-TQFT for cobordisms with embedded ribbon graphs. The construction uses that the ribbon graphs are coloured in a modular category $\mathcal{V}$ and meet the boundary of the cobordisms in distinguished arcs. These lie in the components of the boundary and are themselves $\mathcal{V}-$coloured (compatibly with the incident bands of the embedded ribbon graph) and signed. The disinguished arcs are joined in a $\mathcal{V}$-coloured coupon. The decoration gives rise to a form of gluing that is parameterised in decorated surfaces.

4.1 3-Cobordism Theories with Decorations

We consider decorated cobordisms, where the arcs in the boundary are coloured with objects of $\mathcal{V}$. The colouring of the embedded ribbon graphs shall be compatible with the colouring of the arcs in the boundary. This will allow us to describe the properties of the so decorated cobordisms entirely by the topological properties of the underlying spaces and the colouring of the decorations.

4.1.1 Decorated Surfaces

Definition 69. A marked arc is a simple arc α with an object V of $\mathcal{V}$ and a sign $v \in -1, +1$. A closed connected orientable surface is called decorated if it is oriented and comes with a countable set of distinguished marked arcs. A non-connected surface is decorated, if all of its components are. We call decorated surfaces also d-surfaces. Homeomorphisms between d-surfaces that in addition preserve the decoration are called decorated homeomorphisms, or d-homeomorphisms for short.

Definition 70. The inverse$-\Sigma$ of a connected d-surface Σ is obtained by reversing the orientation of Σ and inverting the signs of all marked arcs lying on on Σ. Inversion extends to non-connected surfaces in the obvious way.

4.1.2 Decorated Types

Decorated types abstract decorated surfaces.

Definition 71. We define a decorated type t to consist of a natural number g, the genus, and an m-tuple of marked arcs $\langle W_i, v_i \rangle_{i \leq m}$. We briefly call decorated types d-types.

Definition 72. The d-type of a d-surfaces of genus g and with m marked arcs $\langle W_i, v_i \rangle_{i \leq m}$ is a tuple $\langle g; \langle W_1, v_1 \rangle, ..., \langle W_m, v_m \rangle \rangle$.

Definition 73. The inverse $-t$ of a decorated type t is obtained by inverting all signs.

Each d-type t gives rise to a canonical d-surfaces Σ_t as follows.

Definition 74. The canonical surface Σ_t of a d-type $t = \langle g; \langle V_1, v_1 \rangle, ..., \langle V_m, v_m \rangle \rangle$ is constructed as follows. Let R_t be the ribbon graph with one coupon and $m+g$ bands. The first$m$ bands are untwisted and unlinked. For $i \leq m$, the i-th band is coloured in the respective label V_i. The sign determines the orientation of the band. In addition there are g bands that with the coupon form unknots.

We then fix a closed regular neighbourhood U_t of R_t. Then U_t is a handlebody of genus g. R_t lies in the interior of U_t, except for the m bands that meet the boundary ∂U_t. The desired d-surface is then $\Sigma_t := \partial U_t$.

The reader should not have problems verifying the soundness of the construction above.

Exercise 75. Show that Σ_t is of d-type t.

The following is easy to see, given that t and $-t$ differ only in the signs. We leave the proof to the reader.

Exercise 76. Prove that $\Sigma_{-t} = -\Sigma_t$ are homeomorphic.

Definition 77. A connected d-surface Σ of d-type t is said to be parameterised if it is homeomorphic to Σ_t. The homeomorphism $\Sigma_t \to \Sigma$ is called the parameterisation of Σ.

4.2 A 3D-TQFT for 3-Cobordism Theories with Decorations

4.2.1 Construction of a Modular Functor from Decorations

Each d-type $t = \langle g; \langle W_1, v_1 \rangle, ..., \langle W_m, v_m \rangle \rangle$ defines a projective K-module Ψ_t as follows:

$$\Phi(t; i) = W_1^{v_1} \otimes ... \otimes W_m^{v_m} \otimes \bigotimes_{r=1}^{g} (V_{i_r} \otimes V_{i_r}^*) \tag{4.1}$$

where $i = (i_1, ..., i_g) \in I^g$, and then set

$$\Psi_t = \bigoplus_{i \in I^g} Hom(\mathbb{I}, \Phi(t; i)) \tag{4.2}$$

Then define $\mathcal{T}(\Sigma)$ to be the tensor product of Ψ_t for types t of the components of Σ, thus for connected Σ of type$t$ we get $\mathcal{T}(\Sigma) = \Psi_t$. The action $\mathcal{T}$ on d-homeomorphisms f is defined by the parameterisations. For connected Σ and Σ', we thus obtain $\mathcal{T}(f)$ to be the identity morphism. Otherwise $\mathcal{T}(f)$ is the tensor product of identity morphisms by restriction to connected components. The proof of the following is immediate from the above construction.

Lemma 78. *$\mathcal{T}$ is a modular functor.*

Below we need an endomorphism $\eta(\Sigma) : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ for each closed oriented surface Σ. η is defined on the summands $Hom(\mathbb{I}, \Phi(t; i))$ by multiplication with $(rank(\mathcal{V}))^{1-g} \prod_{n=1}^{g} dim(i_g)$, and on non-connected surfaces such that $\eta(\Sigma_1 \coprod \Sigma_2) = \eta(\Sigma_1) \otimes \eta(\Sigma_2)$ and $\eta(\emptyset) = id_K$.

4.2.2 Operator Invariants of Decorated 3-Cobordisms

We define an operator invariant for decorated 3-cobordisms M roughly as follows. First we glue standard handlebodies to the boundary of M, which yields a closed oriented 3-manifolds $\widetilde{M}$ with embedded ribbon graph $\widetilde{\Omega}$. The colouring of the extension of $\widetilde{\Omega}$ over Ω is not unique. Fixing a colouring we can apply the topological invariant of ribbon graphs developed in Section **??** to $(\widetilde{M}, \widetilde{\Omega})$. This yields an element of K. This assignment is polylinear in the coupons on the boundary of M, and thus yields a K-homomorphism:

$$\mathcal{T}(\partial_- M) \otimes_K (\mathcal{T}(\partial_+ M))^* \to K \tag{4.3}$$

The action of $\tau(M)$ is then defined as the composition of the adjoint transpose $\mathcal{T}(\partial_- M) \to \mathcal{T}(\partial_+ M)$ with the endomorphism $\eta(\partial_+ M) : \mathcal{T}(\partial_+ M) \to \mathcal{T}(\partial_+ M)$ constructed above.

4.3 Soundness

In the remainder of this section we outline cornerstones of the proof that the construction yields a non-degenerate TQFT.

Theorem 79. *$\langle \mathcal{T}, \tau \rangle$ is a non-degenerate 3D-TQFT.*

4.3.1 Outline

In Lemma... we have shown $\mathcal{T}$ to be a modular functor, so that it remains to verify the following axioms of Definition 48.

- (naturality) follows from τ being a topological invariant of $\langle \widetilde{M}, \widetilde{\Omega} \rangle$.
- (multiplicativity) follows from the multiplicativity of τ with respect to closed 3-manifolds and ultimately from the multiplicativity of the functor $F : Rib_{\mathcal{V}} \to \mathcal{V}$.

- We prove (functoriality) and (normalisation) below. The proofs require a representation of 3-cobordisms by ribbon graphs.

Lemma 80. *(normalisation)* $\tau(\Sigma \times [0,1]) = id_{\mathcal{T}(\Sigma)}$ *for any d-surface* Σ.

Lemma 81. *(functoriality) Let a decorated 3-cobordism* M *be obtained by glueing decorated 3-cobordisms* M_1 *and* M_2 *along a d-homeomorphism* $f : \partial_+ M_1 \to \partial_- M_2$, *then* $\tau(M) = k\tau(M_2)\mathcal{T}f\tau(M_2)$.

To show non-degeneracy, we compute τ on the decorated handlebody $H(U_t, R_t, i, x)$ for $x \in Hom(\mathbb{I}, \Phi(t;i))$, as a cobordism between the empty surface and the canonical surface $\partial H(U_t, R_t, i, x) = \Sigma_t$of the d-type t. We see that $\tau(H(U_t, R_t, i, x)) \in \mathcal{T}(\Sigma_t) = \Psi_t$.

Lemma 82. *For any d-type* t *of genus* g *and any* $x \in Hom(\mathbb{I}, \Phi(t;i))$ *with* $i \in I^g$, *we have* $\tau(H(U_t, R_t, i, x)) = x$.

4.3.2 Representation of 3-Cobordisms by Ribbon Graphs

To show (normalisation) and (functoriality) one needs to represent cobordisms M with embedded ribbon graphs and by special ribbon graphs Ω. Then one performs surgery along Ω in $S^3$and obtains a cobordism of the same d-types and genus. The latter can be shown by first attaching coupons to the free ends of the embedded graph and then using Theorem 38 . The preserved properties suffice for the presentation of 3-cobordisms by ribbon graphs.

One then uses that τ and F coincide on M and Ω, respectively. The properties (normalisation) and (functoriality) then follow from the functoriality of F.

Chapter 5

Anomalies

5.1 Anomaly 2-Cocycles

The (functoriality) axioms of Definition 48 tells us that each pair of cobordisms M_1 and M_2, and homeomorphism $f : \partial_+(M_1) \to \partial_-(M_2)$ give rise to gluing anomalies. We study cases where anomalies yield so-called 2-cocycles of gluing patterns in abelian groups. In these cases we may replace the underlying cobordism theory by one determined by the cocycle.

Definition 83. Let G be an abelian group. A G-valued 2-cocycle g assigns to each gluing pattern an element of G, such that

1. (naturality) g identifies homeomorphic gluing patterns.
2. (multiplicativity) Let P be the disjoint union of gluing patterns P_1 and P_2, then $g(P) = g(P_1)g(P_2)$.
3. (normalisation) g takes gluing with a cylinder to 1, the unit of G.
4. (compatibility) with disjoint union:

$$g(M \coprod N, W, e \coprod f) = g(M, (W, W_M, \partial_+ W \coprod -W_N), e) g(N, (M \cup_e W, W_N, -\partial_- M \coprod \partial_+ W), f) \tag{5.1}$$

where $W = W_M \coprod W_N$

Exchanging the roles of M and N in (compatibility) one finds

$$g(M, \langle W, W_M, \partial_+ W \coprod -W_N), e\rangle g\langle N, \langle M \cup_e W, W_N, -\partial_- M \coprod \partial_+ W\rangle, f\rangle = g\langle N, \langle W, W_N, \partial_+ W \coprod -W_M\rangle, f\rangle$$

The latter necessitates the notion of symmetric gluing patterns, which identifies a gluing patter $\langle M, N, f\rangle$ with its opposite

$$\langle\langle N, -\partial_+ N, -\partial_- N\rangle, \langle M, -\partial_+ M, -\partial_- M\rangle, -f^{-1} : -\partial_- N \to -\partial_+ M\rangle \tag{5.2}$$

Symmetric 2-cocycles satisfy the following variant (compatibility)' of the above axiom.

$$g\langle W, M \coprod N, e \coprod f\rangle = g\langle\langle W, \partial_- W \coprod -Y, X\rangle, M, e\rangle g\langle\langle W \cup_e M, \partial_- W \coprod -\partial_+ M, Y\rangle, N, f\rangle \tag{5.3}$$

5.2 Renormalisation of Cobordisms by means of 2-Cocycles

In the following let G be an abelian group and g a G-valued 2-cocycle.

Definition 84. The cobordism M weighted in$G$ is a pair $\langle M, k\rangle$ where $k \in G$ is called the multiplicative weight of M.

1. Homeomorphisms between G-weighted spaces are homeomorphisms between the underlying spaces that preserve the weight.
2. The weights are compatible with disjoint union, such that$\langle M, k\rangle \coprod \langle N, l\rangle = \langle M \coprod N, kl\rangle$.
3. The boundaries of weighted cobordisms are the boundaries of the underlying cobordisms.

Gluing of weighted cobordisms is defined by gluing cylinders to weighted cobordisms.

Definition 85. Consider a cobordism M whose boundary is a disjoint union of surfaces X, Y, and Z, and let $f : X \to -Y$ be a parameterising homeomorphism. Then we define the associated gluing pattern $P(M, X, Y, Z, f) = \langle\langle X \times [0,1]\rangle, \emptyset, (-X \times 0) \coprod (X \times 1)\rangle, \langle M, -X \coprod -Y, Z\rangle, id_{-X} \coprod f\rangle$. The gluing pattern evaluates to $\langle M', kg(P(M, X, Y, Z, f)\rangle$ where M' is the result of gluing X to Y along f.

Proposition 86. *Weighted cobordisms with weighted gluing form a cobordism theory.*

5.3 Anomaly Cocycles of 3D-TQFTs, Anomaly-free Renormalisation

Let $\langle \mathcal{T}, \tau\rangle$ be a 3D-TQFT with ground ring K. Let G be the abelian group if invertible elements of K. An anomaly cocycle of $\langle \mathcal{T}, \tau\rangle$ is a symmetric G-valued 2-cocycle g of the underlying cobordism theory.

We shall assume that all cobordisms at hand have collars, or otherwise the weight is stable under gluing cylinders to weighted cobordisms.

Example 87. The trivial cocycle is an anomaly cocycle in any anomaly-free TQFT.

Let g be an anomaly cocycle for$\langle \mathcal{T}, \tau\rangle$, we define a renormalised anomaly-free TQFT $\langle \mathcal{T}, \tau^g\rangle$ where

$$\tau^g\langle M, k\rangle := k^{-1}\tau(M) : \mathcal{T}(\partial_- M) \to \mathcal{T}(\partial_+ M)). \tag{5.4}$$

The so obtained TQFT is anomaly free.

Theorem 88. *Let $\langle \mathcal{T}, \tau\rangle$be a TQFT and g an anomaly cocycle of $\langle \mathcal{T}, \tau\rangle$then the renormalisation $\langle \mathcal{T}, \tau^g\rangle$ is an anomaly free TQFT.*

Example 89. The TQFT of Example 51 has anomaly cocycles.

Proposition 90. *Renormalising TQFTs preserves non-degeneracy.*

Anomaly cocycles for TQFTs based on involutive cobordism theories are anomaly cocycles for the dual TQFT.

5.4 Lagrangian Spaces, Lagrangian Relations and Maslov Indices: Preliminary Definitions

5.4.1 Lagrangian Spaces and Contractions

Definition 91. A symplectic vector space is a finite dimensional real vector-space H with an antisymmetric bilinear form $w : H \times H \to \mathbb{R}$. The same vector space with the opposite form will be denoted $-H$. Let $A \subseteq H$ be linear subspace of H, an annihilator of A is an element $h \in H$ such that $w(h, a) = 0$. Denote by $Ann(A)$ the set of annihilators of A. A lagrangian subspace is the maximal linear subspace $A \subseteq H$ such that $A \subseteq Ann(A)$. The latter condition is called isotropy. The set of lagrangian subspaces of H is denoted as $\Lambda(H)$.

Definition 92. Let $A \subseteq H$ be an isotropic subspace, we call $\lambda \mid A$ the lagrangrian contraction of a linear subspace $\lambda \subseteq H$ along A

$$\lambda \mid A = (\lambda + A) \cap Ann(A)/A \subseteq H \mid A \tag{5.5}$$

5.4.2 Lagrangian Relations and Actions Thereof

Definition 93. A lagrangian relation between non-degenerate symplectic vector spaces $H_1$and $H_2$is a lagangian subspace N of $-H_1 \oplus H_2$. We denote lagrangian relations like this one as $N : H_1 \Rightarrow H_2$.

Lagrangian relations generalise symplectic relations.

Exercise 94. Show that lagrangian relations are

1. **reflexive,** such that $diag(H)$ is a lagrangian relation
2. **symmetric**, such that whenever $N : H_1 \Rightarrow H_2$, then $N_s := (h_2, h_1) \mid (h_1, h_2) \in N$ is a lagrangian relation $H_2 \Rightarrow H_1$
3. and **transitive**, such that whenever $N_{12} : H_1 \Rightarrow H_2$and $N_{23} : H_2 \Rightarrow H_3$ are lagrangian relations, then so is $N_{12} \oplus N_{23} : H_1 \Rightarrow H_3$

5.4.3 Maslov Indices

Subsequently let $\lambda_1, \lambda_2, \lambda_3$be isotropic subspaces of a symplectic vector space $\langle H, w \rangle$. We define a bilinear form $\langle \_, \_ \rangle$ on $(\lambda_1 \oplus \lambda_2) \cap \lambda_3$such that for all $a, b \in (\lambda_1 \oplus \lambda_2) \cap \lambda_3$with $a = a_1 + a_2$.

$$\langle a, b \rangle = w(a_2, b) \tag{5.6}$$

Note that a_2 is determined by a up to the addition by a_1. $\langle \_, \_ \rangle$ is well-defined because the possible $a_1$annihilate b.

Exercise 95. Verify that $\langle \_, \_ \rangle$ is symmetric.

Note that $\langle _, _ \rangle$may be degenerate as its annihilator contains $(\lambda_1 \cap \lambda_2) + (\lambda_2 \cap \lambda_3)$.

Definition 96. Define the Maslov index $\mu(\lambda_1, \lambda_2, \lambda_3)$ to be the signature of $\langle \_, \_ \rangle$, that is the number of positive and negative entries.

Proposition 97. *The Maslov index is antisymmetric, in particular we have for any isotropic spaces*

$$\mu(\lambda_1, \lambda_2, \lambda_3) = -\mu(\lambda_2, \lambda_1, \lambda_3) = -\mu(\lambda_1, \lambda_3, \lambda_2) \tag{5.7}$$

5.5 A Digression into Homologies of Surfaces

Let Σ be an oriented surface, we define the real vector space $H_1(\Sigma; \mathbb{R})$. The existence of this homology is a corollary of the Poincare Duality Theorem (see [2] for details).

5.5.1 The Lagrangian Functor

We define a covariant functor from the category of decorated surfaces and 3-cobordisms to the category of non-degenerate symplectic vector spaces with fixed Lagrangian subspaces and Lagrangian relations. This functor assigns to each d-surfaces Σ a lagrangian subspace of its homologies, $H_1(\Sigma; \mathbb{R})$, and to each decorated 3-cobordism a lagrangian relation between homologies of its bases.

Let Σ be a connected parameterised d-surface, so that there is a parameterising d-homeomorphism $\Sigma \to \Sigma_t$. We define a distinguished Lagrangian subspace $\lambda(\Sigma)$ as the kernel of the inclusion map $H_1(\Sigma_t; \mathbb{R}) \to H_1(U_t; \mathbb{R})$. Then for Σ set $\lambda(\Sigma) = f_*[\lambda(\Sigma_t)]$. The distinguished Lagrangian space for disconnected surfaces Σ is defined as the Lagrangian subspace of $H_1(\Sigma, \mathbb{R})$ generated from the distinguished Lagrangian subspaces of its components.

For any decorated 3-cobordism M we have

$$H_1(\partial M; \mathbb{R}) = -H_1(\partial_- M; \mathbb{R}) \oplus H_1(\partial_+ M; \mathbb{R}) \tag{5.8}$$

Taking the kernel of the inclusion morphism $H_1(\partial M; \mathbb{R}) \to H_1(M; \mathbb{R})$ yields a Lagrangian relation $N(M) : H_1(\partial_- M; \mathbb{R}) \Rightarrow H_1(\partial_+ M; \mathbb{R})$.

Remark 98. The definition of $N(M)$ is independent from the decoration.

Exercise 99. Verify covariance, in particular for M being the result of gluing cobordisms M_1 and M_2 along a homeomorphism $p : \partial_+ M_1 \to \partial_- M_2$, then $N(M) = N(M_2) p_* N(M_1)$.

5.6 Computing Anomalies

Theorem 100. *Let M be obtained by gluing decorated 3-cobordisms $M_1$and M_2 along a d-homeomorphism $p : \partial_+ M_1 \to \partial_- M_2$ commuting with the parameterisations, then*

$$\tau(M) = (rank(\mathcal{V})\Delta^{-1})^{\mu(p_*(N_1)_*(\lambda_-(M_1)), \lambda_-(M_2), N_2^*(\lambda_+(M_2)))} \tau(M_2)(\mathcal{T}p)\tau(M_1) \tag{5.9}$$

and for both $i \in 0, 1$

$$N_i = N(M_i) : H_1(\partial_- M_i; \mathbb{R}) \Rightarrow H_1(\partial_+(M_i); \mathbb{R}) \tag{5.10}$$

Remark 101. The gluing anomalies are completely determined by topology of the underling manifold under gluing and by the parameterisations of their bases.

5.7 The Anomaly 2-Cocycle

We define actions of a modular groupoid which determine the anomaly 2-cocycle.

Definition 102. The mapping class group Mod_g is the group of isotopy classes of (degree-1) homeomorphisms on a closed oriented surface of genus g. This definition extends to decorated types t, yielding Mod_t.

A weak homeomorphisms as degree-1 homeomorphisms preserving the decoration up to a permutation of arcs. Then the definition of the mapping class group extends to the groupoid of weak homeomorphisms. A weak homeomorphism $g : \Sigma \to \Sigma'$ defines a cobordism $M(g)$ as the cylinder $\Sigma' \times [0, 1]$ with the top base decorated by $id_{\Sigma'}$ and the bottom base is decorated and parameterised by g. We obtain an action $\tau(M(g))$ for g of the mapping class groupoid with the following value.

Theorem 103. *Let $\Sigma, \Sigma', \Sigma''$ be closed oriented surfaces and $id_\Sigma : \Sigma \to \Sigma$, $g : \Sigma \to \Sigma'$, and $h : \Sigma' \to \Sigma''$ be weak homeomorphisms, then we obtain by Theorem 100*

$$\tau(M(id_\Sigma)) = id_{\mathcal{T}(\Sigma)} \tau(M(gh)) = (rank(\mathcal{V})\Delta^{-1})^{\mu(h_*(\lambda(\Sigma)),\lambda(\Sigma'),g_*^{-1}(\lambda(\Sigma'')))} \tau(M(g))\tau(M(h)) \tag{5.11}$$

where we use that $\mu((gh)_*(\lambda_t), g_*(\lambda_t), \lambda_t) = \mu(h_*(\lambda_t), \lambda_t, g^{-1}(\lambda_t))$.

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