Infinite Traces through Coalgebras

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Overview

Some Motivating Memes

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1. We give a generic definition of infinite trace semantics.
2. We characterise the codomain of infinite trace semantics.
3. We show that the acceptance behaviour of coalgebra automata is infinite trace semantics.

A Review of Finite Trace Semantics

Ingredients A monad B, a functor T, and a (B, T)-coalgebra $\gamma : X \to BTX$

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2. $\overline{\gamma} : X \to \overline{T}X$ is a $\overline{T}$-coalgebra in $Kl(B)$
3. Assume, we find a map $tr_0 : X \to \emptyset$ in $Kl(B)$
4. Define generic finite trace semantics by induction

$$\begin{tikzcd}
 & X \arrow[r, "\overline{\gamma}"] \arrow[dl, bend right, "tr_0"'] \arrow[dr, bend left, "tr_n"] \arrow[drr, bend left, "tr_{n+1}"] & \overline{T}X \arrow[dr, bend left, "\overline{T}tr_n"] & & \\
\emptyset & \cdots & \overline{T}^n\emptyset & \overline{T}^{n+1}\emptyset & \cdots
\end{tikzcd} \tag{1}$$

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Observation That $cod(tr_0) = \emptyset$ means that there are no successfully terminating traces of depth 0.

Generic Infinite Trace Semantics

Idea We start induction with $tr_0^\infty := F\{x \mapsto * \mid x \in X\}$, meaning that by looking at all traces of depth 0 descending from x we do not know which (possibly) non-terminating traces there are.

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Definition $(tr_n^\infty : X \to \overline{T}^n\{*\})_{n<\omega}$ is the generic infinite trace semantics of a (B, T)-coalgebra $\gamma : X \to BTX$.

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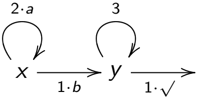
Generic Infinite Trace Semantics: Example

Example Consider the (B, T)-coalgebra where $B = \left(\mathbb{N}^{(-)}\right)_{\omega}$ and $T(-) = \{\surd\} + (-) + \{a, b\} \times (-)$



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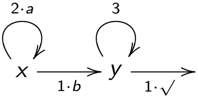

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Compute the generic infinite trace semantics tr^∞ of γ

1. $tr_0^\infty(x) = tr_0^\infty(y) = *$
2. $tr_1^\infty(x) = 2 \cdot a(tr_0^\infty(x)) + 1 \cdot b(tr_0^\infty(y)) = 2 \cdot a * +1 \cdot b*$
   $tr_1^\infty(y) = 3 \cdot (tr_0^\infty(y)) + \surd = 3 \cdot * + 1 \cdot \surd$
3. $tr_2^\infty(x) = 2 \cdot a(tr_1^\infty(x)) + 1 \cdot b(tr_1^\infty(y)) = 4 \cdot aa * +2 \cdot ab * +3b * +1b\surd$
   $tr_2^\infty(y) = 3 \cdot (tr_1^\infty(y)) + \surd = 9 \cdot * + 4 \cdot \surd$
4. ...

Branching and Forgetting

- There is a natural forgetful map $\overline{T}^n F!_{T\{*\}} : \overline{T}^{n+1}\{*\} \to \overline{T}^n\{*\}$.
- $FSeq^T = (\overline{T}^n\{*\}, \overline{T}^n F!_{F\{*\}})_{n<\omega}$ is the lifted final T-sequence.

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& & X \arrow[r, "\overline{\gamma}"] \arrow[d, "tr_n^\infty"'] & \overline{T}X \arrow[d, "\overline{T}tr_n^\infty"] & \\
\{*\} & \cdots \arrow[l, "F!_{T\{*\}}"'] & \overline{T}^n\{*\} \arrow[l] & \overline{T}^{n+1}\{*\} \arrow[l, "\overline{T}^n F!_{T\{*\}}"'] & \cdots \arrow[l]
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Observation For all depths $n < \omega$, pointwise $tr_n^\infty \leq \overline{T}^n F!_{T\{*\}} \circ \overline{T}\, tr_n^\infty \circ \overline{\gamma}$ where $\leq$ is generated from $\leq_{\mathbb{N}}$

Cauchy Sequences: An Example

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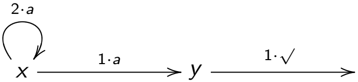

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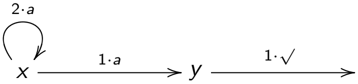

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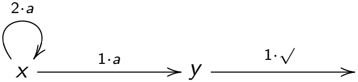

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Note

1. $tr_0^{\infty}(x) \ \leq \ F!_{T\{*\}} tr_1^{\infty}(x) = 3 \cdot *$
2. $tr_1^{\infty}(x) \ \leq \ \overline{T} F!_{T\{*\}} tr_2^{\infty}(x) = 7 \cdot a*$
3. $tr_2^{\infty}(x) \ \leq \ \overline{T}^2 F!_{T\{*\}} tr_3^{\infty}(x) = 14 \cdot aa * + 1 \cdot a\surd$
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Continuations in Spans

Define the iterated distributive law $(\pi^n : T^n B \Rightarrow B T^n)_{n<\omega}$ as

$$\pi^1 = \pi, \ \pi^{n+1} = \pi^n_T \circ T^n \pi \tag{6}$$

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Define the spans $(\leq_n \subseteq BT^n\{*\} \times BT^n\{*\})_{n<\omega}$

$$\begin{tikzcd}
BT^nBT\{*\} \arrow[r, "B\pi^n_{T\{*\}}"] \arrow[d, "BT^n!_{BT\{*\}}"'] & BBT^nT\{*\} \arrow[r, "\mu_{T^nT\{*\}}"] & BT^nT\{*\} \arrow[d, "BT^n!_{T\{*\}}"] \\
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Example for $n = 2$

$$\begin{tikzcd}
\begin{array}{c} 4aa(2a*+1a*)+ \\ 2aa(1\surd)+1a\surd \end{array} \arrow[r, mapsto] \arrow[d, mapsto] & \begin{array}{c} 4*2aaa*4*1aaa*+ \\ 2*1aa\surd+1a\surd \end{array} \arrow[r, mapsto] & \begin{array}{c} 12aaa*+ \\ 2aa\surd+1a\surd \end{array} \arrow[d, mapsto] \\
6aa*+1a\surd & \leq_2 & 14aa*+1a\surd
\end{tikzcd}$$

Infinite Trace Semantics

Definition A sequence in $w \in \prod_{i<\omega} \overline{T}^{i}\{*\}$ is increasing if for all $i < \omega$,

$$w(i) \leq_n \overline{T}^{n} F!_{T\{*\}}(w(i+1)) \tag{7}$$

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Theorem

1. *Generic infinite trace semantics is an infinite trace semantics.*
2. *Finite trace semantics is an infinite trace semantics.*

Trace Semantics from T-Behaviour

To verify that something is an infinite trace semantics, one needs a map turning the BT-behaviour into trace semantics at an arbitrary finite depth.

Intuition This is because, $\overline{T}^n F!_{T\{*\}}$ forgets "at the end", whereas $\overline{\gamma}$ adds "to the front".

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$$\theta^{n+1} : (BT)^n BT \xrightarrow{\theta^n} BT^n BT \xrightarrow{B\pi^n_T} BBT^n T \xrightarrow{\mu_{T^{n+1}}} BT^n T \tag{9}$$

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We obtain trace semantics of depth n from BT-behaviour of depth n.

Bootstrapping a Justification

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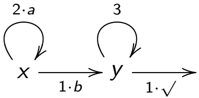
Proof.

$$\begin{tikzcd}
X \arrow[d, "\gamma^n"] \arrow[rd, bend left, "\gamma^{n+1}"] & & & & \\
(BT)^n X \arrow[r, "(BT)^n\gamma"] \arrow[d, "\theta^n_X"] & (BT)^n(BT)X \arrow[d, "\theta^n_{BTX}"] \arrow[drr, bend left, "\theta^{n+1}"] & & & \\
BT^n X \arrow[r, "(BT^n)\gamma"] \arrow[d, "(BT^n)!_X"] & BT^n BTX \arrow[r, "B\pi^n_{TX}"] \arrow[d, "(BT^n)(BT)!_X"] & BBT^n TX \arrow[r, "\mu_{T^n TX}"] \arrow[d, "BBT^n T!_X"] & BT^n TX \arrow[d, "BT^n T!_X"] & \\
BT^n 1 & BT^n BT1 \arrow[l, "BT^n!_{BT1}"'] \arrow[r, "B\pi^n_{T1}"] & BBT^n T1 \arrow[r, "\mu_{T^n T1}"] & BT^n T1 \arrow[r, "(BT^n)!_{T1}"] & BT^n 1
\end{tikzcd}$$

□

Seeing the Forest despite the Böhm Trees

Example Recall the (B, T)-coalgebra γ



(10)

and the infinite trace semantics of γ at y

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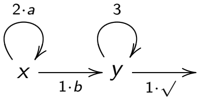

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5. ...

A Böhm-tree is an increasing sequence $(w(i))_{b \leq i < \omega}$ where $0 \leq b$ and $w(i) \in \overline{T}^b\{*\} \setminus \overline{T}^b\emptyset$ for all $i \geq b$.

Note that in our example Böhm trees arise from silent transitions, only.

Coalgebra Automata: Preliminaries in One Slide

Syntax A coalgebra automaton $\mathbb{A}$ is a pointed $(\mathcal{P}, T)$-coalgebra $\langle Q, \tau : Q \to \mathcal{P}TQ, q_I \rangle$ with a finite state set Q, and an acceptance condition induced by a ranking function $\Omega : Q \to \mathbb{N}$.

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Acceptance Games Whether $\mathbb{A}$ accepts a pointed coalgebra $\mathbb{S} = \langle S, \sigma : S \to TS, s_I \rangle$ is defined in terms of a 2-player graph game $\mathcal{G}(\mathbb{A}, \mathbb{S})$.

Position	Player	Sets of admissible moves	$\Omega_{\mathcal{G}}$
$(q, x) \in Q \times S$	$\exists$	$\{(a, x) \mid a \in \theta(q)\}$	$\Omega(q)$
$(a, x) \in TQ \times S$	$\exists$	$\{Z \subseteq Q \times S \mid (a, \tau(x)) \in RelT(Z)\}$	0
$Z \subseteq Q \times S$	$\forall$	$\{(q', x') \in Q \times S \mid (q', x') \in Z\}$	0

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Winning Condition $\exists$ wins a finite play if $\forall$ gets stuck in the last position and an infinite play p if the largest priority of an automaton state occuring infinitely often in a position from $Q \times S$ is even. $\forall$ wins all other plays.

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Acceptance Condition $\mathbb{A}$ accepts $\mathbb{S}$ if $\exists$ has a winning strategy in $G(\mathbb{A}, \mathbb{S})$ from (q_I, s_I). Abstracting from q_I, gives a relation $Acc_\Omega : Q \to \mathcal{P}S$

Acceptance Behaviour as a Coalgebra Map

Lemma

$Acc_\Omega : Q \to S$ is a $\overline{T}$-coalgebra morphism for all T-coalgebras $\sigma : S \to S$

$$\begin{tikzcd}
Q \arrow[r, "Acc_\Omega"'] \arrow[d, "\overline{\tau}"'] & S \arrow[d, "F\sigma"] \\
\overline{T}Q \arrow[r, "\overline{T}Acc_\Omega"] & \overline{T}S
\end{tikzcd} \tag{11}$$

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Proof.

In particular for every $q \in Q$ and $s \in Acc_\Omega(q)$, $F(s) \in \overline{T}(\tau(q))$. The basic position (q, s) is winning for $\exists$ iff all successor positions from some Z are winning, where $(\Phi, \sigma(s)) \in Rel\,T$ and $\Phi \in \tau(q)$. Thus if T is standard, $F\sigma(s) \in \bigcup\{TAcc_\Omega(\Phi) \mid \Phi \in \tau(q)\}$ which is equivalent to the commutation of the above diagram. ☐

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Note This holds in particular for the final T-coalgebra $\xi : Z \to TZ$.

Acceptance Behaviour as Trace Semantics

Theorem

Acc_{Ω} is an infinite trace semantics for all Ω.

Define $\Omega_0(q) := 0$ and $\Omega_1(q) := 1$ for all $q \in Q$, then **(1)** *Acc^{Ω_0} yields finite trace semantics and* **(2)** *Acc^{Ω_1} yields generic infinite trace semantics of $\tau : Q \to \mathcal{P}TQ$.* **(3)** *$Acc^{\Omega_1}$ is Jacob's infinite trace semantics.*

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Proof.

To show Acc_Ω an infinite trace semantics one uses that Acc_Ω is a $\overline{T}$-coalgebra map that composes with the generic infinite trace semantics tr^∞ of the final T-coalgebra $\xi : Z \to TZ$. The latter commutes with the Seq^T.

$$\begin{tikzcd}
Q \arrow[r, "Acc_\Omega"'] \arrow[d, "\overline{\tau}"'] & Z \arrow[r, "tr_n^\infty"'] \arrow[d, "F\xi"] & \overline{T}^n\{*\} \\
\overline{T}Q \arrow[r, "\overline{T}Acc_\Omega"] & \overline{T}Z \arrow[r, "\overline{T}tr_n^\infty"] & \overline{T}^{n+1}\{*\} \arrow[u, "\overline{T}^n F!_{T\{*\}}"']
\end{tikzcd} \tag{12}$$

A proof of (3) uses determinisation of non-deterministic coalgebra automata. ☐

- **Jacobs**, *Trace Semantics for Coalgebras*, 2004

Where to go from here?

Infinite Trace Semantics

- The codomain $cod(tr)$ of an infinite trace semantics is a (non-free) B-algebra.
- $cod(tr)$ is a coalgebra for small-enough T and B.
- Characterise $cod[Tr^{\infty}_{\gamma}]$ topologically.

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Trace Logics

- Infinite trace semantics provides invariants; algebraise them.
- Expand previous work on trace logics via concrete dualities.

Thank You

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References

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- **Kissig, Kurz**, *Generic Trace Logics*, to appear

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