# Trace Logics for Semiring Monads

Christian Kissig
(Joint Work with Alexander Kurz)

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# Overview

1. Motivating Example: Coalgebra Automata
2. Finitary Trace Semantics
3. Semiring Monads
4. Categories of Semimodules
5. Trace Logics for Semiring Monads

# Deterministic Automata

Deterministic Automata on Finite Words

- set $Q$ of states, $q_I \in Q$ initial
- a transition map $\tau : Q \times A \to 1 + Q$.

**Semantics**

$$Acc(q, (at)) := \begin{cases} true & \text{if } \tau(q, a) = * \\ Acc(q', t) & \text{where } q' = \tau(q, a) \end{cases}$$

**Notation** $1 = \{*\}$

# Nondeterministic Automata

Nondeterministic Automata on Finite Words

- set $Q$ of states, $q_I \in Q$ initial
- a transition map $\tau : Q \times A \to 1 + \mathcal{P}(Q)$

**Semantics**

$$Acc(q, (at)) := \begin{cases} \mathit{true} & \text{if } \tau(q, a) = * \\ Acc(q', t) & \text{for some } q' \in \tau(q, a) \end{cases}$$

**Notation** $1 = \{*\}$

# Shortsighted Coalgebra Automata

Coalgebra Automata $\mathbb{A}$ on Finite Words

- $\mathcal{S}et$-functor $\mathcal{F} := 1 + A \times (-)$
- pointed coalgebra $\langle Q, \tau : Q \to \mathcal{P}\mathcal{F}Q, q_I \rangle$

**Literature** Venema, *Automata and Fixed Point Logics: A Coalgebraic Perspective*, 2006

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$\mathcal{F}$ admits a final coalgebra $\langle A^*, \xi : A^* \to 1 + A \times A^* \rangle$ defined as

$$\xi(at) := (a, t)$$

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**Semantics** of $\mathbb{A}$

$$\begin{array}{ccc} Q & \xrightarrow{\quad \tau \quad} & 1 + A \times Q \\ {\scriptstyle tr_\tau} \downarrow & & \downarrow {\scriptstyle \overline{\mathcal{F}} tr_\tau} \\ A^* & \xrightarrow[\quad \xi \quad]{} & 1 + A \times A^* \end{array}$$

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# Trace Semantics for Coalgebras

We wish to define a trace semantics uniformly for

- Coalgebras $\langle X, \sigma : X \to \mathcal{P}\mathcal{F}X \rangle$
- $\mathcal{S}et$-functors $\mathcal{F}$
- Monads $\langle \mathcal{T}, \mu, \eta \rangle$

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# Trace Semantics for Coalgebras

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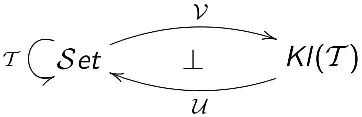
# The Kleisli-Category

Every monad $\langle \mathcal{T}, \mu, \eta \rangle$ admits an adjunction




**defined** such that

- $\mathcal{V}(X) := X$, $\mathcal{V}(f : X \to Y) := \eta \circ f$
- $\mathcal{U}(X) := \mathcal{T}X$, $\mathcal{U}(f : A \to B) := \mu \circ \mathcal{T}f$

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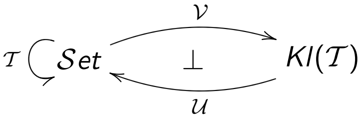


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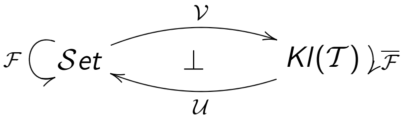
**Then** $\mathcal{T} = \mathcal{U}\mathcal{V}$

# Lifting Functors into $Kl(\mathcal{T})$



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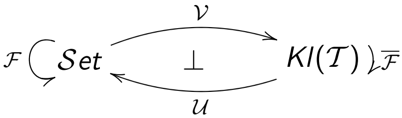


A distributive law is a natural transformation $\pi : \mathcal{F}\mathcal{T} \Rightarrow \mathcal{T}\mathcal{F}$

- $\pi \circ \mathcal{F}\eta = \eta_{\mathcal{F}}$
- $\pi \circ \mathcal{F}\mu = \mu_{\mathcal{F}} \circ \mathcal{T}\pi \circ \pi_{\mathcal{T}}$

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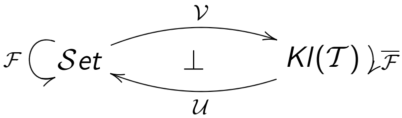


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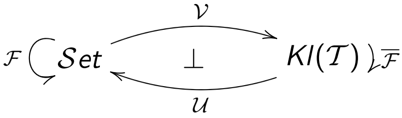
**Example** Let $\mathcal{T} = \mathcal{P}$ and $\mathcal{F} = 1 + A \times (-)$, then $\pi : \mathcal{F}\mathcal{T} \Rightarrow \mathcal{T}\mathcal{F}$ is defined such that

$$\pi(*) := \{*\} \text{ and } \pi(a, Y \subseteq X) := \{(a, y) \mid y \in Y\}$$

# Lifting Functors into $Kl(\mathcal{T})$




**Define** $\overline{\mathcal{F}}$ from $\mathcal{F}$ and $\pi : \mathcal{F}\mathcal{T} \Rightarrow \mathcal{T}\mathcal{F}$ such that

- on objects, $\overline{\mathcal{F}}(X) := \mathcal{F}(X)$
- on morphisms, $\overline{\mathcal{F}}(f : X \to Y) := \pi \circ \mathcal{F}(f)$

We use that morphisms $f : X \to Y$ in $Kl(\mathcal{T})$ are essentially morphisms $f : X \to \mathcal{T}Y$ in $\mathcal{S}et$

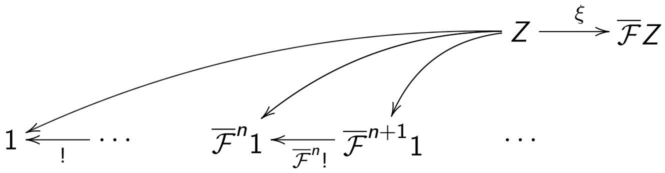
# Final Coalgebra Semantics in $Kl(\mathcal{T})$ by Final Sequence Induction

$$1 \xleftarrow{\;!\;} \cdots \qquad \overline{\mathcal{F}}^{n}1 \xleftarrow{\;\overline{\mathcal{F}}^{n}!\;} \overline{\mathcal{F}}^{n+1}1 \qquad \cdots$$

**Literature** Jacobs, *Trace Semantics for Coalgebras*, 2004

# Final Coalgebra Semantics in $\mathcal{Kl}(\mathcal{T})$ by Final Sequence Induction

**Assume** that the final $\overline{\mathcal{F}}$-sequence terminates in $\omega$ steps.

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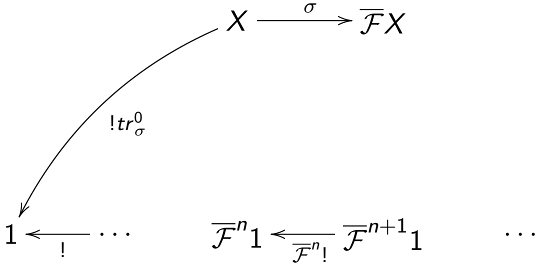


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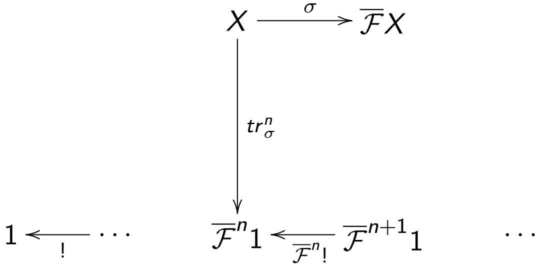


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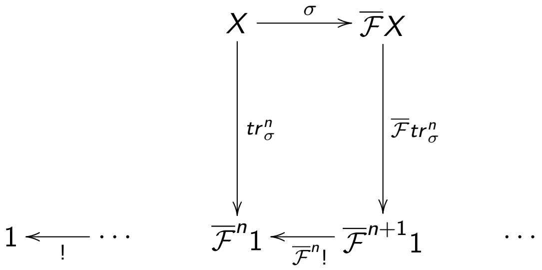


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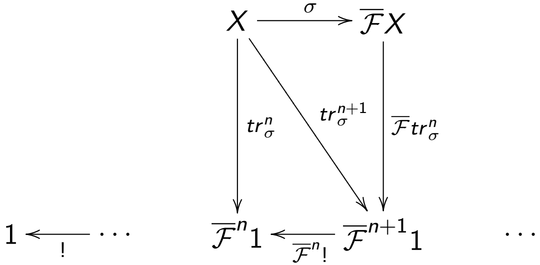


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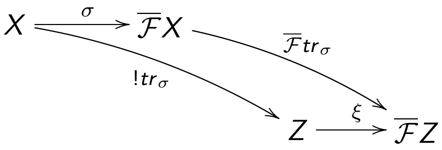


$$1 \xleftarrow{\;!\;} \cdots \qquad \overline{\mathcal{F}}^n 1 \xleftarrow{\;\overline{\mathcal{F}}^n !\;} \overline{\mathcal{F}}^{n+1} 1 \qquad \cdots$$

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# Termination of the Final Sequence

- for most $\mathcal{S}et$-functors $\mathcal{F}$, the initial $\mathcal{F}$-sequence has a limit in $\mathcal{S}et$
- the limit of the inital $\mathcal{F}$-sequence is preserved by $\mathcal{V} : \mathcal{S}et \to Kl(\mathcal{T})$
- limit-colimit-coincidence yields colimit of the final $\overline{\mathcal{F}}$-sequence in $Kl(\mathcal{T})$

**Literature** Hasuo, Jacobs, Sokolova, *Generic Trace Theory*, 2006

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**Assumption**

- $Kl(\mathcal{T})$ can be enriched over a $DCPO_{\perp}$

**Literature** Smyth, Plotkin, *The Category Theoretic Solution to Recursive Domain Equations*, 1977

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**Assumption**

- $Kl(\mathcal{T})$ can be enriched over a $DCPO_{\perp}$

**More General Assumption**

- $Kl(\mathcal{T})$ is locally chain complete

**Literature** Fiore, Cattani, *The Bicategory Theoretic Solution to Recursive Domain Equation*, 2007

# Semirings and Semimodules

**Definition** A semiring is a structure $\mathcal{S} = \langle S, +, *, 0, 1 \rangle$

- $\langle S, +, 0 \rangle$ is a commutative monoid
- $\langle S, *, 1 \rangle$ is a monoid
- $x * (y + z) =_{\mathcal{S}} (x * y) + (x * z)$

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**Examples**

- $2 = \langle \{0, 1\}, \vee, \wedge, 0, 1 \rangle$
- $\mathbb{N} = \langle \mathbb{N}, +, *, 0, 1 \rangle$ is a semiring
- $\mathbb{R}_+^\infty = \langle \mathbb{R}_+^\infty, +, min, 0, \infty \rangle$

# Semimodules

**Definition** A left $\mathcal{S}$-semimodule is a structure
$_{\mathcal{S}}M = \langle M, \oplus, 0_M, (s \cdot (-) : M \to M)_{s \in S} \rangle$

- $\langle M, \oplus, 0_M \rangle$ is a commutative monoid
- $(x * y) \cdot m = x \cdot (y \cdot m)$
- $0 \cdot m = 0_M$

**Example** $\mathcal{S}$ is a left $\mathcal{S}$-semimodule $\langle S, +, 0, (s * (-))_{s \in S} \rangle$.

# Semimodules

**Definition** A right $\mathcal{R}$-semimodule is a structure $M_{\mathcal{R}} = \langle M, \oplus, 0_M, ((-) \cdot r) : M \to M)_{r \in R} \rangle$

- $\langle M, \oplus, 0_M \rangle$ is a commutative monoid
- $m \cdot (x * y) = (m \cdot x) \cdot y$
- $m \cdot 0 = 0_M$

**Example** $\mathcal{S}$ is a right $\mathcal{S}$-semimodule $\langle S, +, 0, ((-) * s)_{s \in S} \rangle$.

# Semiring Monads

Map $X \mapsto \left(\mathcal{S}^X)\right)_\omega$ extends to $\mathcal{S}et$-functor with

- $\left(\mathcal{S}^{f:X \to Y})\right)_\omega (m \in \left(\mathcal{S}^X)\right)_\omega) := \lambda y \in Y. \sum_{x | f(x) = y} m(x)$

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Monad on $(\mathcal{S}^{(-)}))_\omega$

- $\eta(x \in X) := \lambda x'. \begin{cases} 1 & x = x' \\ 0 & \text{otherwise} \end{cases}$
- $\mu(n \in (\mathcal{S}^{(\mathcal{S}^X))_\omega}))_\omega) := \lambda x. \sum_{m \in (\mathcal{S}^X))_\omega} n(m) * m(x)$

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**Remark** Algebras for $\left(\mathcal{S}^{(-)})\right)_\omega$ are (left) $\mathcal{S}$-semimodules.

# Bisemimodules

**Definition** $\mathcal{S}$-$\mathcal{R}$-bisemimodules are structures

$$_{\mathcal{S}}M_{\mathcal{R}} = \langle U, \oplus, 0_U, (s \cdot (-))_{s \in S}, ((-) \cdot r)_{r \in R} \rangle$$

- $_{\mathcal{S}}U = \langle U, \oplus, 0_U, (s \cdot (-))_{s \in S} \rangle$ is a left $\mathcal{S}$-semimodule
- $U_{\mathcal{R}} = \langle U, \oplus, 0_U, ((-) \cdot r)_{r \in R} \rangle$ is a right $\mathcal{R}$-semimodule

**Example** $\mathcal{S} = \langle S, +, *, 0, 1 \rangle$ is an $\mathcal{S}$-$\mathcal{S}$-bisemimodule

$$\mathcal{S} = \langle S, +, 0, (s * (-))_{s \in S}, ((-) * s)_{s \in S} \rangle$$

# Morita Functors

$$ {}_{\mathcal{S}}SMod \underset{{}_{\mathcal{S}}Hom(-,{}_{\mathcal{S}}U_{\mathcal{R}})}{\overset{Hom_{\mathcal{R}}(-,{}_{\mathcal{S}}U_{\mathcal{R}})}{\rightleftarrows}} SMod_{\mathcal{R}}^{op} $$

# Morita Functors

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$$

**Define** $Hom_{\mathcal{R}}(-, {}_{\mathcal{S}}U_{\mathcal{R}})$ and ${}_{\mathcal{S}}Hom(-, {}_{\mathcal{S}}U_{\mathcal{R}})$ on objects as

- $Hom_{\mathcal{R}}({}_{\mathcal{S}}M, {}_{\mathcal{S}}U_{\mathcal{R}})$ is the right $\mathcal{R}$-semimodule

$$\langle Hom({}_{\mathcal{S}}M, {}_{\mathcal{S}}U), \oplus_U, 0_U, ((-) \cdot r)_{r \in R} \rangle$$

- $Hom_{\mathcal{R}}(N_{\mathcal{R}}, {}_{\mathcal{S}}U_{\mathcal{R}})$ is the left $\mathcal{S}$-semimodule

$$\langle Hom(N_{\mathcal{R}}, U_{\mathcal{R}}), \oplus_U, 0_U, (s \cdot (-))_{s \in R} \rangle$$

# Morita Adjunction

**Proposition**

$$
{}_{\mathcal{S}}SMod \underset{{}_{\mathcal{S}}Hom(-,{}_{\mathcal{S}}U_{\mathcal{R}})}{\overset{Hom_{\mathcal{R}}(-,{}_{\mathcal{S}}U_{\mathcal{R}})}{\rightleftarrows}} SMod_{\mathcal{R}}^{op} \qquad \bot
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# Morita Adjunction

**Proposition**

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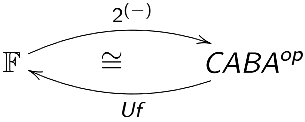
**Proof** The isomorphism

$$ \frac{{}_{\mathcal{S}}SMod({}_{\mathcal{S}}M, {}_{\mathcal{S}}Hom(N_{\mathcal{R}}, {}_{\mathcal{S}}U_{\mathcal{R}}))}{SMod_{\mathcal{R}}(N_{\mathcal{R}}, Hom_{\mathcal{R}}({}_{\mathcal{S}}M, {}_{\mathcal{S}}U_{\mathcal{R}}))} $$

is given by the curry isomorphism.

# Example: Stone Duality

Stone duality between finite sets and complete atomic boolean algebras




is a special case of Morita duality where

$$\mathcal{S} = \mathcal{R} = \langle \{0,1\}, \vee, \wedge, 0, 1 \rangle \text{ and } {}_{\mathcal{S}}U_{\mathcal{R}} = \mathcal{S}$$

# Morita Adjunction, a Duality?

**Theorem** There is no duality between ${}_{\mathcal{S}}SMod$ and $SMod_{\mathcal{R}}$ for any $\mathcal{S}$ and $\mathcal{R}$.
**Proof Idea** Limits and colimits do not distribute.

**Literature** Anderson, Fuller, *Rings and Categories of Modules*, 1992

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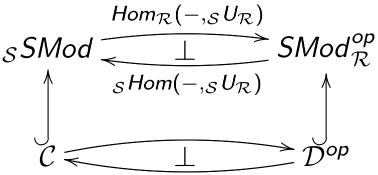
**Proposition** The Morita adjunction on the categories of finitely freely generated left and right $\mathcal{S}$-semimodules is a dual equivalence.
**Proof Idea**

- finitely freely generated semimodules are finite copowers of the zero semimodule
- finite products and coproducts coincide in categories of semimodules

**Literature** Anderson, Fuller, *Rings and Categories of Modules*, 1992

# Excursion: Morita Duality




**Literature** Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, 1958
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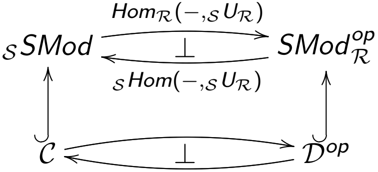


**Assumptions**

- ${}_{\mathcal{S}}U_{\mathcal{R}}$ is a balanced bisemimodule
- ${}_{\mathcal{S}}U$ and $U_{\mathcal{R}}$ are injective cogenerators of $\mathcal{C}$ and $\mathcal{D}$ respectively

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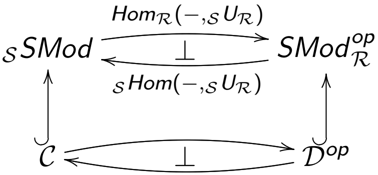


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**Theorem** (Morita) Then, $Hom_{\mathcal{R}}(-, {}_{\mathcal{S}}U_{\mathcal{R}}) \dashv_{\mathcal{S}} Hom(-, {}_{\mathcal{S}}U_{\mathcal{R}})$ is an equivalence between $\mathcal{C}$ and $\mathcal{D}^{op}$.

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# Morita Dual Functors

$$\overline{\mathcal{F}} \circlearrowright {}_{\mathcal{S}}FSMod \underset{{}_{\mathcal{S}}Hom(-,{}_{\mathcal{S}}U_{\mathcal{R}})}{\overset{Hom_{\mathcal{R}}(-,{}_{\mathcal{S}}U_{\mathcal{R}})}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} FSMod^{op}_{\mathcal{R}} \circlearrowleft \underline{\mathcal{F}}$$ [1]

**Define** $\underline{\mathcal{F}}(-) := Hom_{\mathcal{R}}(\overline{\mathcal{F}}{}_{\mathcal{S}}Hom(-,{}_{\mathcal{S}}U_{\mathcal{R}}), {}_{\mathcal{S}}U_{\mathcal{R}})$

[1] ${}_{\mathcal{S}}FSMod$ and $FSMod_{\mathcal{R}}$ are the categories of free left $\mathcal{S}$- and right $\mathcal{R}$-semimodules

# Morita Dual Functors

$$\overline{\mathcal{F}} - CoAlg \underset{\perp}{\rightleftarrows} (\underline{\mathcal{F}} - Alg)^{op}$$

$$\overline{\mathcal{F}} \circlearrowright {}_{\mathcal{S}}FSMod \underset{{}_{\mathcal{S}}Hom(-,{}_{\mathcal{S}}U_{\mathcal{R}})}{\overset{Hom_{\mathcal{R}}(-,{}_{\mathcal{S}}U_{\mathcal{R}})}{\underset{\perp}{\rightleftarrows}}} FSMod_{\mathcal{R}}^{op} \circlearrowleft \underline{\mathcal{F}}$$ [1]

**Define** $\underline{\mathcal{F}}(-) := Hom_{\mathcal{R}}(\overline{\mathcal{F}}_{\mathcal{S}}Hom(-, {}_{\mathcal{S}}U_{\mathcal{R}}), {}_{\mathcal{S}}U_{\mathcal{R}})$

Then $Hom_{\mathcal{R}}(-, {}_{\mathcal{S}}U_{\mathcal{R}}) \dashv {}_{\mathcal{S}}Hom(-, {}_{\mathcal{S}}U_{\mathcal{R}})$ extends to a dual adjunction between the categories of $\overline{\mathcal{F}}$-coalgebras over ${}_{\mathcal{S}}FSMod$ and $\underline{\mathcal{F}}$-algebras over $FSMod_{\mathcal{R}}$.

[1] ${}_{\mathcal{S}}FSMod$ and $FSMod_{\mathcal{R}}$ are the categories of free left $\mathcal{S}$- and right $\mathcal{R}$-semimodules

# Trace Logics via Morita Adjunctions

**Assumption** ${}_{\mathcal{S}}FSMod$ and $FSMod_{\mathcal{R}}$ are locally chain complete.

# Trace Logics via Morita Adjunctions

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- $Hom_{\mathcal{R}}(-, \mathcal{S})$ is left adjoint, and thus preserves colimits (of the initial sequence)
- transpose of the final $\overline{\mathcal{F}}$-coalgebra under the Morita adjunction is the initial $\underline{\mathcal{F}}$-algebra

# Trace Logics via Morita Adjunctions

$$\overline{\mathcal{F}} \circlearrowright {}_{\mathcal{S}}FSMod \qquad\qquad FSMod_{\mathcal{S}} \circlearrowleft \underline{\mathcal{F}}$$

$$Z \xrightarrow[\xi]{} \overline{\mathcal{F}}Z \qquad\qquad A \xleftarrow[Hom_{\mathcal{R}}(\xi,\mathcal{S})]{} \underline{\mathcal{F}}A$$

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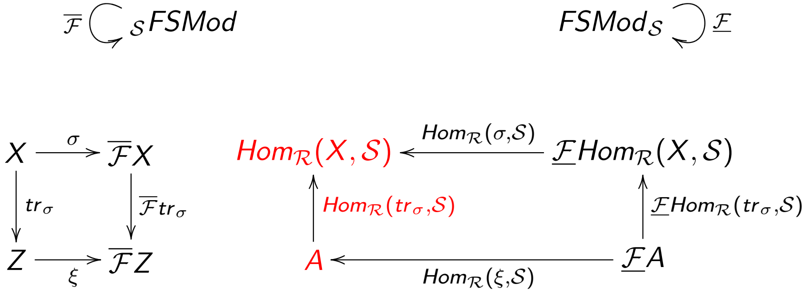


**Define**

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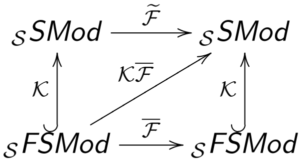
- and semantics in $\langle X, \sigma \rangle$ as $Hom_{\mathcal{R}}(tr_\sigma, \mathcal{S})$

# Extending Functors




**Define** $\widetilde{\mathcal{F}}$ as the left Kan extension $Lan_{\mathcal{K}}\mathcal{K}\overline{\mathcal{F}}$, concretely

$$\widetilde{\mathcal{F}}(-) := \int_{\mathcal{S}}^{\mathcal{S}\langle X\rangle} SMod({}_{\mathcal{S}}\langle X\rangle, -) \cdot (\mathcal{K}\overline{\mathcal{F}}_{\mathcal{S}}\langle X\rangle)$$

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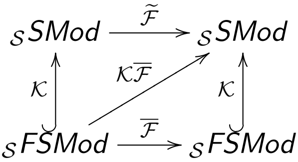


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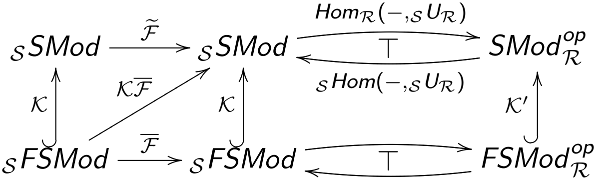


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# Full Trace Logics

$$
\begin{array}{ccc}
{}_{\mathcal{S}}SMod & \qquad\qquad & SMod_{\mathcal{S}} \\
\mathcal{K} \big\uparrow & & \mathcal{K}' \big\uparrow \\
{}_{\mathcal{S}}FSMod & & FSMod_{\mathcal{S}}
\end{array}
$$

$$
\begin{array}{ccc}
X & \xrightarrow{\sigma} & \overline{\mathcal{F}}X \\
{\scriptstyle tr_\sigma}\big\downarrow & & \big\downarrow \\
Z & \xrightarrow[\xi]{} & \overline{\mathcal{F}}Z
\end{array}
\qquad
\begin{array}{ccc}
Hom_{\mathcal{R}}(X,\mathcal{S}) & \xleftarrow{Hom_{\mathcal{R}}(\sigma,\mathcal{S})} & \underline{\mathcal{F}}Hom_{\mathcal{R}}(X,\mathcal{S}) \\
\big\uparrow {\scriptstyle Hom_{\mathcal{R}}(tr_\sigma,\mathcal{S})} & & \big\uparrow \\
A & \xleftarrow[Hom_{\mathcal{R}}(\xi,\mathcal{S})]{} & \underline{\mathcal{F}}A
\end{array}
$$

[2]

[2] $\widehat{\mathcal{F}}$ is the Morita dual of $\widetilde{\mathcal{F}}$

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$$\begin{array}{ccc} {}_{\mathcal{S}}SMod & \qquad\qquad & SMod_{\mathcal{S}} \\ \uparrow{\scriptstyle\mathcal{K}} & & \uparrow{\scriptstyle\mathcal{K}'} \\ {}_{\mathcal{S}}FSMod & & FSMod_{\mathcal{S}} \end{array}$$

$$\begin{array}{ccc} \mathcal{K}X & \xrightarrow{\mathcal{K}\sigma} & \widetilde{\mathcal{F}}\mathcal{K}'X \\ {\scriptstyle\mathcal{K}tr_\sigma}\downarrow & & \downarrow \\ \mathcal{K}Z & \xrightarrow[\xi]{} & \widetilde{\mathcal{F}}\mathcal{K}Z \end{array} \qquad \begin{array}{ccc} \mathcal{K}'Hom_{\mathcal{R}}(X,\mathcal{S}) & \xleftarrow{\mathcal{K}'Hom_{\mathcal{R}}(\sigma,\mathcal{S})} & \widehat{\mathcal{F}}\mathcal{K}'Hom_{\mathcal{R}}(X,\mathcal{S}) \\ \uparrow{\scriptstyle\mathcal{K}'Hom_{\mathcal{R}}(tr_\sigma,\mathcal{S})} & & \uparrow \\ \mathcal{K}'A & \xleftarrow[\mathcal{K}'Hom_{\mathcal{R}}(\xi,\mathcal{S})]{} & \widehat{\mathcal{F}}\mathcal{K}'A \end{array}$$

2

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\end{array}
\qquad\qquad
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\mathcal{K}' \big\uparrow \\
FSMod_{\mathcal{S}}
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{K}X & \xrightarrow{\mathcal{K}\sigma} & \widetilde{\mathcal{F}}\mathcal{K}'X \\
\Big\downarrow \mathcal{K}tr_{\sigma} & & \Big\downarrow \\
\mathcal{K}Z & \xrightarrow[\xi]{} & \widetilde{\mathcal{F}}\mathcal{K}Z \\
\Big\downarrow ! & & \Big\downarrow \\
{}_{\mathcal{S}}Z & \xrightarrow[\zeta]{} & \widetilde{\mathcal{F}}_{\mathcal{S}}Z
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{K}'Hom_{\mathcal{R}}(X,\mathcal{S}) & \xleftarrow{\mathcal{K}'Hom_{\mathcal{R}}(\sigma,\mathcal{S})} & \widehat{\mathcal{F}}\mathcal{K}'Hom_{\mathcal{R}}(X,\mathcal{S}) \\
\Big\uparrow \mathcal{K}'Hom_{\mathcal{R}}(tr_{\sigma},\mathcal{S}) & & \Big\uparrow \\
\mathcal{K}'A & \xleftarrow[\mathcal{K}'Hom_{\mathcal{R}}(\xi,\mathcal{S})]{} & \widehat{\mathcal{F}}\mathcal{K}'A \\
\Big\uparrow ! & & \Big\uparrow \\
A_{\mathcal{S}} & \xleftarrow[Hom_{\mathcal{R}}(\zeta,\mathcal{S})]{} & \widehat{\mathcal{F}}A_{\mathcal{S}}
\end{array}
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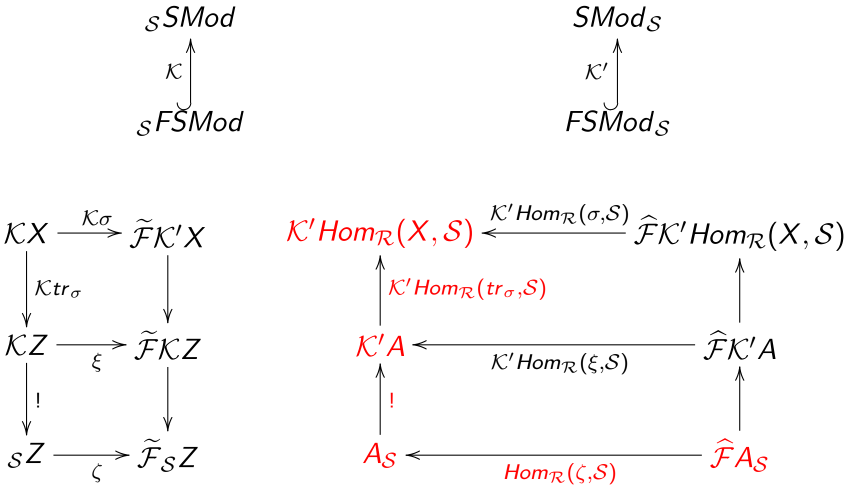


2

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# Adequacy

**Theorem** The semantics of trace logics is invariant under coalgebra-morphisms.

**Proof** follows from the definition of trace logics.

# Expressivity

**Theorem** Points not trace-equivalent can be distinguished by trace logics.

**Proof** follows from the definition of trace logics and the property of the final coalgebra map in $\mathcal{S}et$ to be mono.

# Conclusions

### Summary

- We have defined a coalgebraic logic adequate and expressive up to trace equivalence for a relevant class of monads.
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We are grateful to Bart Jacobs for various hints and helpful discussions which sparked many of the ideas in this work.

**Thank You**